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Constructing solutions of Hamilton–Jacobi equations for 2 D fields with one component by means of Bäcklund transformations

Wulf Böttger¹, Henning Wissowski², Hans A. Kastrup³
Institut für Theoretische Physik E, RWTH Aachen,
D-52056 Aachen, Fed. Rep. Germany

Abstract

The Hamilton–Jacobi formalism generalized to 2-dimensional field theories according to Lepage’s canonical framework is applied to several relativistic real scalar fields, e.g. massless and massive Klein–Gordon, Sinh & Sine–Gordon, Liouville and ϕ^4 theories. The relations between the Euler–Lagrange and the Hamilton–Jacobi equations are discussed in DeDonder and Weyl’s and the corresponding wave fronts are calculated in Carathéodory’s formulation. Unlike mechanics we have to impose certain integrability conditions on the velocity fields to guarantee the transversality relations and especially the dynamical equivalence between Hamilton–Jacobi wave fronts and families of extremals embedded therein. Bäcklund Transformations play a crucial role in solving the resulting system of coupled nonlinear PDEs.

1 Introduction

Varying a relativistically invariant action integral leads to covariant Euler–Lagrange equations. However, if one wants to reformulate the theory in terms of the conventional canonical Hamiltonian framework one has to break the manifest covariance by distinguishing a time variable and regarding the other ”spatial” coordinates as ”indices” representing an infinite number of degrees of freedom. The method is widely known from elementary particle physics, canonical gravity and other field theories. This approach, however, can obscure a part of the rich geometrical structure contained in a generally covariant framework, at least on the classical level [1]:

¹e-mail: wulf@thphys.physik.rwth-aachen.de

²e-mail: henning@acds16.physik.rwth-aachen.de

³e-mail: kastrup@thphys.physik.rwth-aachen.de

Utilizing Cartan's theory of alternating forms Lepage and others showed that a large variety of algebraically inequivalent covariant Hamiltonian formulations, including Hamilton–Jacobi equations, exists, e.g. that of DeDonder & Weyl [10] [11] and that of Carathéodory [12], where only the latter one provides a Hamilton–Jacobi equation the associated wave fronts of which have the same nice transversality properties with respect to the extremals as one has in mechanics. Reformulations and special examples of this covariant Hamiltonian approach in terms of multisymplectic frameworks may be found in [1][2][3][5].

In mechanics one can construct solutions of the canonical eqs. of motion if one has an appropriate solution of the corresponding Hamilton–Jacobi equation, the solutions of which describe wave fronts which are transversal to a "field" of extremals and which contain the same dynamical information as the extremals themselves.

For field theories this is no longer true [1]: Solutions of the Hamilton–Jacobi equation (HJE) associated with one of the canonical frameworks mentioned above provide "velocity" fields which in general do not obey the necessary integrability conditions (IC). The latter have to be postulated separately and give rise to an additional set of partial differential equations. However, if these equations and the associated Hamilton–Jacobi equation are satisfied, then combined they contain the same dynamical content as the Euler–Lagrange equations [1]. Thus, it is possible to construct solutions of the Euler–Lagrange field equations (ELE) by solving a Hamilton–Jacobi equation and a set of integrability conditions.

The aims of the present paper are the following ones:

1. All the different canonical formalisms just mentioned coincide if the field has just one real component [1], especially for fields in 2 spacetime or Euclidean dimensions. In this case one can use the more simple DeDonder & Weyl HJE in order to solve it simultaneously with one integrability condition. Here we do not construct completely new solutions of the ELE but find one-parameter extensions of a given solution φ_0 . The method is applied to the following one-component models in 2 dimensions: massive and massless Klein–Gordon, Sinh-, Sine–Gordon, Liouville and ϕ^4 theory. The solutions of the HJE plus IC are constructed by expanding the solutions of these equations in powers of the field variables. This leads to a hierarchy of nonlinear PDEs that can be transformed into linear PDEs with nonconstant coefficients. By applying integrable Bäcklund transformations these PDEs can be reduced further to free (massive or massless) Klein–Gordon equations. Remarkably, it is only necessary to solve just two linear PDEs in order to obtain the general solution for *every* order of the hierarchy!
2. The solutions of the DeDonder & Weyl HJE do not have the appropriate transversality properties required to construct the wave fronts associated with a given one-parameter set of extremals. Those have to be given in terms of the solutions of the more complicated HJE of Carathéodory. As we are dealing with one-component fields here only, the two canonical frameworks are equivalent and one can construct the 1–dimensional Carathéodory wave fronts from the solutions of the DeDonder & Weyl HJE. This procedure will be outlined and applied to given solutions (e.g. solitons) of models mentioned above.

The paper is organized as follows:

In chapter 2 we very briefly summarize Lepage's reformulation of canonical mechanics. In chapter 3 we apply this framework to field theories with one component in two dimensions and present the HJEs of DeDonder & Weyl and of Carathéodory, respectively. In chapter 4 we study the hierarchy of equations derived within the Hamilton–Jacobi framework of DeDonder & Weyl and the associated integrability condition for the velocity fields by expanding the wave fronts in powers of the field variable φ in the neighbourhood of φ_0 .

In chapter 5 we determine the Bäcklund transformations which reduce this hierarchy of linear PDEs with nonconstant coefficients to free field equations.

In chapter 6 we apply the general results to several well known models and point out some relations to stability problems of solitons.

In chapter 7 we discuss the relations between the hierarchies of PDEs derived from expanding the HJE and the IC and a corresponding expansion of the Euler–Lagrange equations. Solutions of the equations of motions are determined from those of the HJE and the IC perturbatively. The kink solution of the Sine–Gordon equations is treated in considerable detail here.

In chapter 8 we calculate the wave fronts by means of Carathéodory's formulation, especially for the kink solution just mentioned.

2 Lepage's Canonical Formulation of Mechanics

We very briefly recall Lepage's main idea of introducing the canonical formalism in mechanics for one configuration variable q . The general case is discussed in [1].

The starting point of the canonical theory for Lagrangian canonical systems is an action functional. In mechanics it is given by a Lagrangian 1-form ω integrated along a path $\mathcal{C} := \{t, q = q(t)\}$ in the extended configuration space $\mathcal{M}_{1+1} := \{(t, q)\}$:

$$\mathcal{A}[\mathcal{C}] = \int_{\mathcal{C}} \omega = \int_{\mathcal{C}} L(t, q, \dot{q}) dt. \quad (2.1)$$

As to the variational principle it is preferable to consider the generalized velocity v as an independent variable, which coincides with \dot{q} on the extremals $\mathcal{C} = \mathcal{C}_0 := \{(t, q = q_0(t))\}$. Normally this extension in the number of variables is performed by using Lagrangian multipliers. Lepage's reformulation of the variational principle is similar in spirit [4].

The initial canonical Lagrangian form $\omega = L(t, q, \dot{q}) dt$ is extended by the product of a Lagrangian multiplier $h(t, q, v)$ and the Pfaffian form $\varrho = dq - v dt$ vanishing on the tangent vectors of the extremals, which ensures the identification of $v(t)$ with $\dot{q}(t)$ on the solutions of the equation of motion. Then the action integral $\mathcal{A}[\mathcal{C}]$ over the path $\mathcal{C} := \{(t, q(t))\}$ can be modified:

$$\mathcal{A}[\mathcal{C}] \rightarrow \tilde{\mathcal{A}}[\mathcal{C}] = \int_{\mathcal{C}} \Omega = \int_{\mathcal{C}} [L(t, q, v) dt + h(t, q, v) \varrho], \quad (2.2)$$

without changing the Euler–Lagrange equations and their solutions $\mathcal{C} = \mathcal{C}_0$.

The form ϱ generates an ideal $I[\varrho]$ in the algebra Λ of forms on the extended configuration space $\mathcal{M}_{1+1} := \{(t, q)\}$: if $\alpha \in \Lambda$ and $\beta \in I[\varrho]$, then $\alpha \wedge \beta$ is also an element of the ideal $I[\varrho]$.

The Lagrangian multiplier $h(t, q, v)$ can be fixed by varying the action integral (2.2) with respect to q, v independently. This leads to the standard definition of the canonical momentum: $h \stackrel{!}{=} \partial_v L =: p$.

We obtain the same results by requiring $d\Omega$ to be an element of the ideal $I[\varrho]$:

$$d\Omega = (\partial_v L - h) dv \wedge dt + (dh - \partial_q L dt) \wedge \varrho = \underbrace{(\partial_v L - h)}_{=0!} dv \wedge dt + 0 \pmod{I[\varrho]} . \quad (2.3)$$

Hence $d\Omega$ is a closed two form on families of extremals covering the extended configuration space $\mathcal{M}_{1+1} = \{(t, q)\}$ – or correspondingly – a (Lagrangian) submanifold $\mathcal{Q} := \{t, q, p = \psi(t, q) | (t, q) \in M\}$ of the extended phase space $\mathcal{P}_{2+1} := \{(t, q, p)\}$.

Following Poincaré’s lemma Ω is locally (at least) exact $\Omega = dS(t, q)$.

The Legendre transformation $L \rightarrow H, v \rightarrow p$ can be implemented as a change of basis in the cotangent bundle $\mathcal{T}^*(\mathcal{M}_{1+1})$, $\varrho \rightarrow dq, dt \rightarrow dt$:

$$\Omega = L dt + p \varrho = L dt + p (dq - v dt) = -(pv - L)dt + p dq = -H dt + p dq . \quad (2.4)$$

H denotes the usual Hamiltonian $H = pv - L = H(p, q, t)$.

The existence of a potential $S(t, q)$ for the basic differential form $\Omega = dS$ yields the familiar Hamilton–Jacobi equation for $S(t, q)$ and the corresponding condition for the momentum:

$$\Omega = -H(t, q, p = \psi(t, q)) dt + \psi(t, q) dq \stackrel{!}{=} dS(t, q) = \partial_t S(t, q) dt + \partial_q S(t, q) dq . \quad (2.5)$$

Comparing the coefficients of dt, dq yields:

$$\partial_t S(t, q) + H(t, q, p = \psi(t, q)) = 0, \quad p = \psi(t, q) = \partial_q S(t, q) . \quad (2.6)$$

The extremals can be determined, if a complete integral of the Hamilton–Jacobi equation is found. Dealing with one independent variable q this integral depends on one constant c_0 . The solution of the equation of motion can be calculated by solving the algebraic relation $\partial_{c_0} S(t, q, c_0) = c_1$ with a second constant c_1 .

In mechanics there exists a special relation between the wavefront $S(t, q) = \sigma = \text{const}$, $\sigma \in \mathcal{R}$ and the extremals $q = q_0(t)$ called “*transversality*”: the union of their respective tangent spaces $\text{span}\{e_t = \partial_t + \dot{q}(t)\partial_q\}$ and $\text{span}\{w = p\partial_t + H\partial_q\}$, span the whole tangent space $\mathcal{T}_P(\mathcal{M}_{1+1})$ at any point $P \in \mathcal{M}_{1+1}$, if the Lagrangian L does not vanish, because of

$$\det(e_t, w) = \det \left(\begin{pmatrix} 1 & p \\ v & H \end{pmatrix} \right) = (H - \dot{q}p) = -L .$$

This concept of the Hamilton–Jacobi framework developed for mechanics can easily be generalized to field theories. We confine our discussion to those theories depending on *one* real scalar field $\varphi = \varphi(z, \bar{z})$ in a 1+1 dimensions. Here z, \bar{z} play the role of lightcone variables. For details see [1][10][11].

3 The Hamilton–Jacobi theories of DeDonder & Weyl and of Carathéodory

As in mechanics a canonical theory for fields is based on an action functional $\mathcal{A}[\Sigma]$ defined on a two dimensional surface $\Sigma := \{(z, \bar{z}, \varphi(z, \bar{z}))\}$ in the extended configuration space $\mathcal{M}_{2+1} := \{(z, \bar{z}, \varphi)\}$:

$$\mathcal{A}[\Sigma] = \int_{\Sigma} \omega = \int_{\Sigma} \mathcal{L}(z, \bar{z}, \varphi(z, \bar{z}), v = \partial_z \varphi, \bar{v} = \partial_{\bar{z}} \varphi) dz \wedge d\bar{z}. \quad (3.1)$$

Only on the extremals $\varphi = \varphi_0(z, \bar{z})$ the generalized velocities v, \bar{v} coincide with the derivatives of the fields: $v = \partial_z \varphi_0, \bar{v} = \partial_{\bar{z}} \varphi_0$.

The canonical 2-form $\omega = \mathcal{L} dz \wedge d\bar{z}$ is extended by means of two Lagrangian parameters $h(z, \bar{z}, \varphi), \bar{h}(z, \bar{z}, \varphi)$ and a 1-form $\varrho = d\varphi - v dz - \bar{v} d\bar{z}$ that vanishes on the 2-dimensional extremals $\varphi = \varphi_0(z, \bar{z})$:

$$\Omega = \mathcal{L} dz \wedge d\bar{z} + \bar{h} dz \wedge \varrho + h \varrho \wedge d\bar{z}. \quad (3.2)$$

The Lagrangian multipliers h, \bar{h} are determined by requiring $d\Omega \in I[\varrho]$:

$$d\Omega = (\partial_v \mathcal{L} - h) dv \wedge dz \wedge d\bar{z} + (\partial_{\bar{v}} \mathcal{L} - \bar{h}) d\bar{v} \wedge dz \wedge d\bar{z} + 0 \pmod{I[\varrho]} \stackrel{!}{=} 0 \pmod{I[\varrho]}, \quad (3.3)$$

yielding

$$h \stackrel{!}{=} \partial_v \mathcal{L} =: p, \quad \bar{h} \stackrel{!}{=} \partial_{\bar{v}} \mathcal{L} =: \bar{p}. \quad (3.4)$$

The Legendre transformation $\mathcal{L} \rightarrow \mathcal{H}, \{v, \bar{v}\} \rightarrow \{p, \bar{p}\}$ can be implemented as a change of the basis in the cotangent bundle $\mathcal{T}^*(\mathcal{M}_{2+1})$, $\varrho \rightarrow d\varphi, dz \rightarrow dz, d\bar{z} \rightarrow d\bar{z}$:

$$\Omega = -\mathcal{H} dz \wedge d\bar{z} + \bar{p} dz \wedge d\varphi + p d\varphi \wedge d\bar{z} \quad \text{with} \quad \mathcal{H} := pv + \bar{p}\bar{v} - \mathcal{L}. \quad (3.5)$$

Because $d\Omega = 0 \pmod{I[\varrho]}$ it is locally exact $\Omega = d\mathcal{S}, \mathcal{S} \in \mathcal{T}^*(\mathcal{M}_{2+1})$ on families of extremals. However, contrary to mechanics Ω as an exact 2-form can be represented in different ways by means of a Pfaffian form \mathcal{S} . In the case of DeDonder & Weyl [10][11]:

$$\mathcal{S}_{DW} = S(z, \bar{z}, \varphi) dz - \bar{S}(z, \bar{z}, \varphi) d\bar{z} \quad (3.6)$$

and in the case of Carathéodory [12]:

$$\mathcal{S}_C = S^z(z, \bar{z}, \varphi) dS^{\bar{z}}(z, \bar{z}, \varphi). \quad (3.7)$$

Comparing the exterior derivatives of these expressions with equation (3.5)

$$\Omega = -\mathcal{H} dz \wedge d\bar{z} + \bar{p} dz \wedge d\varphi + p d\varphi \wedge d\bar{z} \stackrel{!}{=} d\bar{S} \wedge dz - dS \wedge d\bar{z}, \quad (3.8)$$

we obtain the Hamilton–Jacobi equations and the transversality conditions for a one component field theory in DeDonder and Weyl’s formulation:

$$\partial_z S + \partial_{\bar{z}} \bar{S} = -\mathcal{H}, \quad p = \partial_{\varphi} S, \quad \bar{p} = \partial_{\varphi} \bar{S} \quad (3.9)$$

and in Carathéodory’s case:

$$\Omega = -\mathcal{H} dz \wedge d\bar{z} + \bar{p} dz \wedge d\varphi + p d\varphi \wedge d\bar{z} \stackrel{!}{=} dS^z \wedge dS^{\bar{z}}, \quad (3.10)$$

we get:

$$\begin{aligned} \partial_z S^z \partial_{\bar{z}} S^{\bar{z}} - \partial_z S^{\bar{z}} \partial_{\bar{z}} S^z &= -\mathcal{H}, \\ p &= \partial_{\bar{z}} S^{\bar{z}} \partial_{\varphi} S^z - \partial_{\bar{z}} S^z \partial_{\varphi} S^{\bar{z}} \\ \bar{p} &= \partial_z S^z \partial_{\varphi} S^{\bar{z}} - \partial_z S^{\bar{z}} \partial_{\varphi} S^z \end{aligned} \quad (3.11)$$

The two theories here are equivalent because a n -form in a space of $n+1$ variables has always rank n [6]. Due to this algebraic equivalence of covariant canonical formulations for one component field theories we may choose DeDonder and Weyl's description to embed the extremals of interest in a system of solutions of the Hamilton–Jacobi equation. We use Carathéodory's theory for the explicit calculation of the wave fronts $S^z = \text{const.}$, $S^{\bar{z}} = \text{const.}$:

In two dimensional field theories involving one field variable the basic two form Ω has always the rank two, i.e. it can be constructed from two independent one forms by linear combination of exterior products. Because Ω is closed its rank is equal to its class, that gives the codimension of the integral submanifold determined by Ω . To calculate this integral manifold — the wavefronts in our case — we can use a corollary of Frobenius' integrability theorem [1]: there exist two functions $S^z(z, \bar{z}, \varphi)$, $S^{\bar{z}}(z, \bar{z}, \varphi)$ such a manner that the exterior product of their differentials equals Ω . The corresponding one dimensional wave fronts are just the submanifolds determined by the conditions $S^z(z, \bar{z}, \varphi) = \text{const.}$, $S^{\bar{z}}(z, \bar{z}, \varphi) = \text{const.}$.

Thus the wave fronts are equipotential surfaces of the solutions of the Hamilton–Jacobi equation formulated in Carathéodory's framework. For simplicity we first solve the HJE of DeDonder & Weyl and the associated IC and afterwards we return to Carathéodory's equation in order to obtain an explicit expression for the wave fronts.

In mechanics for one variable q it is possible to construct wave fronts for 1-parametric families of extremals that cover a certain region of the extended configuration space \mathcal{M}_{1+1} and vice versa. Provided a solution $S(t, q)$ of the Hamilton–Jacobi equation (HJE) is given, the corresponding velocity field, the so-called "slope function":

$$\Phi(t, q) = \partial_p H(t, q, p = \partial_q S(t, q)) \quad (3.12)$$

determines the corresponding 1-parametric extremals by means of the ordinary first order differential equation: $\dot{q}(t) = \Phi(t, q(t))$.

In general this is *not* true for field theories; the ability to embed extremals $\varphi_0(z, \bar{z})$ in a given wave front can be maintained only if the slope functions (velocity fields) $v = \Phi(\varphi, z, \bar{z})$, $\bar{v} = \bar{\Phi}(\varphi, z, \bar{z})$ obtained from the inverse Legendre transformation

$$\begin{aligned} \partial_z \varphi(z, \bar{z}) &= v(p = \partial_{\varphi} S, \bar{p} = \partial_{\varphi} \bar{S}, z, \bar{z}, \varphi) = \Phi(\varphi, z, \bar{z}), \\ \partial_{\bar{z}} \varphi(z, \bar{z}) &= \bar{v}(p = \partial_{\varphi} S, \bar{p} = \partial_{\varphi} \bar{S}, z, \bar{z}, \varphi) = \bar{\Phi}(\varphi, z, \bar{z}) \end{aligned} \quad (3.13)$$

satisfy the integrability condition $\partial_z \partial_{\bar{z}} \varphi(z, \bar{z}) \stackrel{!}{=} \partial_{\bar{z}} \partial_z \varphi(z, \bar{z})$, which results in the requirement

$$\boxed{\frac{d}{d\bar{z}} \Phi(z, \bar{z}, \varphi(z, \bar{z})) := \partial_{\bar{z}} \Phi + \bar{\Phi} \cdot \partial_{\varphi} \Phi = \frac{d}{dz} \bar{\Phi}(z, \bar{z}, \varphi(z, \bar{z})) := \partial_z \bar{\Phi} + \Phi \cdot \partial_{\varphi} \bar{\Phi}} \quad (3.14)$$

on Φ and $\bar{\Phi}$.

Provided the Hamilton–Jacobi equation and the integrability condition are fulfilled, the

Euler–Lagrange equation is satisfied automatically. This can be seen as follows:

Differentiating the Hamilton–Jacobi equation (3.9) with respect to the field variable φ one obtains:

$$\partial_z p + \partial_{\bar{z}} \bar{p} = -\partial_\varphi \mathcal{H} - \partial_p \mathcal{H} \partial_\varphi p - \partial_{\bar{p}} \mathcal{H} \partial_\varphi \bar{p}. \quad (3.15)$$

Because $\partial_\varphi \mathcal{H} = -\partial_\varphi \mathcal{L}$, $\partial_p \mathcal{H} = \Phi$ and $\partial_{\bar{p}} \mathcal{H} = \bar{\Phi}$ (see [1]) we get:

$$\frac{d}{dz} p + \frac{d}{d\bar{z}} \bar{p} = \frac{d}{dz} \frac{\partial \mathcal{L}}{\partial v} + \frac{d}{d\bar{z}} \frac{\partial \mathcal{L}}{\partial \bar{v}} = \partial_\varphi \mathcal{L}. \quad (3.16)$$

The momenta $p(z, \bar{z}, \varphi)$ and $\bar{p}(z, \bar{z}, \varphi)$ are defined on the extended configuration space \mathcal{M}_{2+1} and are considered to be associated with a family of extremals $\varphi = \tilde{\varphi}(z, \bar{z}, u)$ parametrized by u . The Euler–Lagrange equation has to be fulfilled for every single extremal $\varphi = \tilde{\varphi}(z, \bar{z}, u = \text{const.})$. Hence if we insert p and \bar{p} into this equation of motion, we have to be aware of their implicit dependence of z, \bar{z} via the field variable φ . Taking the defining equations (3.13) for the slope functions into account, the total derivatives in the PDEs (3.14) and (3.16) are defined as:

$$\frac{d}{dz} := \left. \frac{\partial}{\partial z} \right|_{\bar{z}, \varphi = \text{const}} + \Phi \left. \frac{\partial}{\partial \varphi} \right|_{\bar{z}, z = \text{const}}, \quad \frac{d}{d\bar{z}} := \left. \frac{\partial}{\partial \bar{z}} \right|_{z, \varphi = \text{const}} + \bar{\Phi} \left. \frac{\partial}{\partial \varphi} \right|_{z, \bar{z} = \text{const}}. \quad (3.17)$$

Therefore the total derivatives are nothing but derivatives with respect to the independent variables z, \bar{z} regarding u to be a constant.

Due to the extend in which the integrability condition is taken into account there exist two different methods of using a Hamilton–Jacobi theory: the weak and the strong embedding of extremals in families of wave fronts.

1.) *Weak embedding:*

This method is used to embed a given single extremal $\hat{\varphi}_0(z, \bar{z})$ in families of wave fronts. In order to obtain a weak embedding it is sufficient to take only the Hamilton–Jacobi equation and the transversality conditions (3.13) on $\hat{\varphi}_0(z, \bar{z})$ into account. In this case usually one chooses a linear ansatz in the field variable for one of the functions S or \bar{S} . However, this approach in general will not provide new extremals, because the IC are fulfilled on the given extremal only. For details to this subject see [19].

2.) *Strong embedding:*

Here one requires the IC not only to hold on the given extremal but in a whole neighbourhood of it. If this is the case then one is able to generate new extremals from a given one by integrating the slope functions (3.13).

In the following we study the strong embedding of a single given extremal into a family of wave fronts.

Like in mechanics there exists a transversality relation between extremals $\varphi(z, \bar{z})$ and wave fronts $S^z(z, \bar{z}, \varphi) = \text{const.}$, $S^{\bar{z}}(z, \bar{z}, \varphi) = \text{const.}$, which holds, if in every point $P \in \mathcal{M}_{1+2}$ the basis of tangent space \mathcal{TM}_{1+2} is given by a union of the basis of the 2-dimensional tangent space of the extremals $e_z = \partial_z + v \partial_\varphi$, $e_{\bar{z}} = \partial_{\bar{z}} + \bar{v} \partial_\varphi$ and a basis vector $w = p \partial_z + \bar{p} \partial_{\bar{z}} + \mathcal{H} \partial_\varphi$ of the 1-dimensional tangent space of the wave fronts, i.e. iff the Lagrangian density \mathcal{L} does not vanish:

$$\det(e_z, e_{\bar{z}}, w) = \det \begin{pmatrix} 1 & 0 & p \\ 0 & 1 & \bar{p} \\ v & \bar{v} & \mathcal{H} \end{pmatrix} = (\mathcal{H} - pv - \bar{p}\bar{v}) = -\mathcal{L} \neq 0. \quad (3.18)$$

Notice that in theories with more than one real field ($d \geq 2$) both \mathcal{L} and \mathcal{H} have to be nonvanishing quantities to guarantee the transversality relation [1].

4 Hamilton–Jacobi theory for one real field in DeDonder and Weyl’s formulation

We here restrict ourselves to Lagrangian densities of the following type: $\mathcal{L} = \partial_z \varphi \partial_{\bar{z}} \varphi - V(\varphi)$. The potential $V(\varphi)$ is an analytic function. Here we have the canonical momenta $p = \bar{v}$, $\bar{p} = v$, the Hamiltonian density $\mathcal{H} = p\bar{p} + V$ and the slope functions $\Phi = \partial_\varphi \bar{S}$, $\bar{\Phi} = \partial_\varphi S$. We have the (DeDonder & Weyl) Hamilton–Jacobi equation

$$\partial_z S + \partial_{\bar{z}} \bar{S} = \partial_\varphi S \partial_\varphi \bar{S} + V(\varphi) \quad (4.1)$$

and the related integrability condition

$$\partial_z \partial_\varphi S - \partial_{\bar{z}} \partial_\varphi \bar{S} = \partial_\varphi S \partial_\varphi^2 \bar{S} - \partial_\varphi \bar{S} \partial_\varphi^2 S. \quad (4.2)$$

Knowing solutions S and \bar{S} of the equations (4.1) and (4.2) a family of embedded extremals $\varphi = \tilde{\varphi}(z, \bar{z})$ is determined by a system of first order PDEs:

$$\partial_z \tilde{\varphi}(z, \bar{z}) = \Phi = \partial_\varphi \bar{S}(z, \bar{z}, \varphi = \tilde{\varphi}), \quad (4.3)$$

$$\partial_{\bar{z}} \tilde{\varphi}(z, \bar{z}) = \bar{\Phi} = \partial_\varphi S(z, \bar{z}, \varphi = \tilde{\varphi}). \quad (4.4)$$

A solution is obtained by expanding $S(z, \bar{z}, \varphi)$ and $\bar{S}(z, \bar{z}, \varphi)$ in powers of the difference $y = \varphi - \varphi_0$ between φ and a known extremal $\varphi_0(z, \bar{z})$:

$$S(z, \bar{z}, \varphi) = \sum_{n=0}^{\infty} \frac{1}{n!} A_n(z, \bar{z}) y^n, \quad \bar{S}(z, \bar{z}, \varphi) = \sum_{n=0}^{\infty} \frac{1}{n!} \bar{A}_n(z, \bar{z}) y^n. \quad (4.5)$$

This method of expanding about a given solution of the equations of motion is commonly employed e.g. with stability investigations or determining (quantum) fluctuations around c-number fields in selfinteracting theories [13] [15].

Naturally φ_0 has to satisfy the transversality relations (4.4), that fix only the functions $A_1 = \partial_{\bar{z}} \varphi_0$ and $\bar{A}_1 = \partial_z \varphi_0$, without influencing the remaining coefficients. Inserting the expressions (4.5) into the HJE (4.1) and expanding the potential V in powers of y we see that the equation is automatically fulfilled up to the order y^1 , whereas the IC (4.2) is fulfilled on the extremals up to the order y^0 .

A_0 and \bar{A}_0 are only affected by the HJE of zeroth order in y :

$$\partial_z A_0 + \partial_{\bar{z}} \bar{A}_0 = \mathcal{L}|_{\varphi=\varphi_0} = \partial_z \varphi_0 \partial_{\bar{z}} \varphi_0 - V(\varphi_0). \quad (4.6)$$

The general solution of the homogeneous equation $\partial_z A_0^{hom} + \partial_{\bar{z}} \bar{A}_0^{hom} = 0$ is given — at least locally — by $A_0^{hom} = \partial_{\bar{z}} \chi_0(z, \bar{z})$ and $\bar{A}_0^{hom} = -\partial_z \chi_0(z, \bar{z})$. One special solution of the inhomogeneous PDE (4.6) can be obtained by integration: $\bar{A}_0^{inh} = \int \mathcal{L}|_{\varphi=\varphi_0} d\bar{z}/2$ and $A_0^{inh} = \int \mathcal{L}|_{\varphi=\varphi_0} dz/2$. This yields the general solution (locally):

$$A_0(z, \bar{z}) = \frac{1}{2} \int [\partial_z \varphi_0 \partial_{\bar{z}} \varphi_0 - V(\varphi_0)] dz + \partial_{\bar{z}} \chi_0(z, \bar{z}), \quad (4.7)$$

$$\bar{A}_0(z, \bar{z}) = \frac{1}{2} \int [\partial_z \varphi_0 \partial_{\bar{z}} \varphi_0 - V(\varphi_0)] d\bar{z} - \partial_z \chi_0(z, \bar{z}). \quad (4.8)$$

In order to determine the coefficients A_2 and \bar{A}_2 , we insert $A_1 = \partial_{\bar{z}}\varphi_0$ and $\bar{A}_1 = \partial_z\varphi_0$ into the IC of order y :

$$\partial_z A_2 - A_3 \partial_z \varphi_0 + \bar{A}_1 A_3 + \bar{A}_2 A_2 = \partial_{\bar{z}} \bar{A}_2 - \bar{A}_3 \partial_{\bar{z}} \varphi_0 + A_1 \bar{A}_3 + A_2 \bar{A}_2. \quad (4.9)$$

Since y is a function of z, \bar{z} , its derivative with respect to z or \bar{z} yields $\partial_z y = -\partial_z \varphi_0$ and $\partial_{\bar{z}} y = -\partial_{\bar{z}} \varphi_0$ respectively. Thus we infer from equation (4.9) that $\partial_z A_2 = \partial_{\bar{z}} \bar{A}_2$ which permits to express these two functions at least locally by one generating potential function: $A_2 = \partial_{\bar{z}} \zeta_2(z, \bar{z})$ and $\bar{A}_2 = \partial_z \zeta_2(z, \bar{z})$. ζ_2 has to fulfil the PDE

$$\partial_{\bar{z}} \partial_z \zeta_2(z, \bar{z}) + \partial_z \zeta_2(z, \bar{z}) \partial_{\bar{z}} \zeta_2(z, \bar{z}) + \frac{1}{2} \left[\partial_\varphi^2 V(\varphi = \varphi_0(z, \bar{z})) \right] = 0. \quad (4.10)$$

The ansatz

$$\zeta_2 = \ln \theta(z, \bar{z}) \quad \Rightarrow \quad A_2 = \partial_{\bar{z}} \ln \theta, \quad \bar{A}_2 = \partial_z \ln \theta \quad (4.11)$$

linearizes the HJE of second order in y :

$$\boxed{\partial_{\bar{z}} \partial_z \theta(z, \bar{z}) + \frac{1}{2} \left[\partial_\varphi^2 V(\varphi = \varphi_0(z, \bar{z})) \right] \theta(z, \bar{z}) = 0.} \quad (4.12)$$

In order to obtain the expressions for A_n and \bar{A}_n one has to substitute the power series (4.5) into the IC and the HJE and to compare the coefficients of the powers y^{n-1} and y^n respectively. Starting with the IC in $(n-1)$ -th order:

$$\begin{aligned} & \partial_z A_n - A_{n+1} \partial_z \varphi_0 + \sum_{p=0}^{n-1} \binom{n-1}{p} \bar{A}_{p+1} A_{n-p+1} \\ &= \partial_{\bar{z}} \bar{A}_n - \bar{A}_{n+1} \partial_{\bar{z}} \varphi_0 + \sum_{p=0}^{n-1} \binom{n-1}{p} A_{p+1} \bar{A}_{n-p+1} \end{aligned} \quad (4.13)$$

we draw our attention to the coefficients A_{n+1} and \bar{A}_{n+1} of highest order. Fortunately they disappear from this equation as well as from the HJE due to the relations $A_1 = \partial_{\bar{z}} \varphi_0$ and $\bar{A}_1 = \partial_z \varphi_0$. Provided the coefficients A_0, \dots, A_{n-1} and $\bar{A}_0, \dots, \bar{A}_{n-1}$ have already been determined recursively, one gets an equation for the functions $A_n(z, \bar{z})$ and $\bar{A}_n(z, \bar{z})$:

$$\begin{aligned} \partial_z (\theta^{n-2} A_n) - \partial_{\bar{z}} (\theta^{n-2} \bar{A}_n) &= \theta^{n-2} \sum_{p=2}^{n-2} \left[\binom{n-1}{p} - \binom{n-1}{p-1} \right] A_{p+1} \bar{A}_{n-p+1} \\ &=: \text{Inh} (A_0, \dots, A_{n-1}; \bar{A}_0, \dots, \bar{A}_{n-1}). \end{aligned} \quad (4.14)$$

The Inhomogeneity $\text{Inh} (A_0, \dots, A_{n-1}; \bar{A}_0, \dots, \bar{A}_{n-1})$ vanishes for $n \leq 4$. Similarly to (4.8) this hierarchy of equations is solved by splitting the solution in two parts: a general solution of the homogeneous part $\partial_z (\theta^{n-2} A_n^{hom}) - \partial_{\bar{z}} (\theta^{n-2} \bar{A}_n^{hom}) = 0$, that is obtained — at least locally — by

$$A_n^{hom} = \theta^{2-n} \partial_{\bar{z}} \zeta_n(z, \bar{z}) \quad \text{and} \quad \bar{A}_n^{hom} = \theta^{2-n} \partial_z \zeta_n(z, \bar{z}) \quad (4.15)$$

with an arbitrary smooth function $\zeta_n(z, \bar{z})$. The second part is a special solution of the inhomogeneous equation (4.14):

$$A_n^{inh} = \theta^{2-n} \partial_{\bar{z}} \bar{\chi}_n(z, \bar{z}) \quad \text{and} \quad \bar{A}_n^{inh} = -\theta^{2-n} \partial_z \bar{\chi}_n(z, \bar{z}) \quad \text{with} \quad (4.16)$$

$$\bar{\chi}_n(z, \bar{z}) = \frac{1}{2} \int^z \int^{\bar{z}} \text{Inh} (A_0, \dots, A_{n-1}; \bar{A}_0, \dots, \bar{A}_{n-1}) \, dz' \, d\bar{z}'. \quad (4.17)$$

The general solution

$$A_n = \theta^{2-n} \partial_{\bar{z}} [\zeta_n(z, \bar{z}) + \bar{\chi}_n(z, \bar{z})] \quad \text{and} \quad \bar{A}_n = \theta^{2-n} \partial_z [\zeta_n(z, \bar{z}) - \bar{\chi}_n(z, \bar{z})] \quad (4.18)$$

has to satisfy the HJE (4.1) to n -th order in y , in which, remarkably, the coefficients of highest order A_{n+1} and \bar{A}_{n+1} vanish as well as in the IC, due to the relations $A_1 = \partial_{\bar{z}} \varphi_0$ and $\bar{A}_1 = \partial_z \varphi_0$. Separating the functions of the remaining highest order A_n , \bar{A}_n leads to the equation:

$$\partial_z (\theta^n A_n) + \partial_{\bar{z}} (\theta^n \bar{A}_n) = -\theta^n \sum_{p=2}^{n-2} \binom{n}{p} \bar{A}_{p+1} A_{n-p+1} - \theta^n \left. \frac{d^n}{d\varphi^n} V(\varphi) \right|_{\varphi=\varphi_0} \quad (4.19)$$

It is convenient to set $\zeta_n(z, \bar{z}) = \theta^{-1} \chi_n$ before inserting into expression (4.19). Using the relation (4.12) the HJE of n -th order in y yields an equation, which is nothing but the inhomogeneous extension of the linear PDE (4.12):

$$\boxed{\partial_z \partial_{\bar{z}} \chi_n + \frac{1}{2} [\partial_{\varphi}^2 V(\varphi_0)] \chi_n = \widetilde{\text{Inh}}(\chi_0, \dots, \chi_{n-1})}, \quad (4.20)$$

with the inhomogeneity:

$$\widetilde{\text{Inh}}(\chi_0, \dots, \chi_{n-1}) = \partial_{\bar{z}} \theta \partial_z \bar{\chi}_n - \partial_z \theta \partial_{\bar{z}} \bar{\chi}_n - \frac{1}{2} \theta^{n-1} \sum_{p=2}^{n-2} \binom{n}{p} \bar{A}_{p+1} A_{n-p+1} - \frac{1}{2} \theta^{n-1} \partial_{\varphi}^n V(\varphi_0). \quad (4.21)$$

In general the coefficients A_n , \bar{A}_n are determined by the n -th order of the HJE and the $n-1$ order of the IC for $n \geq 3$:

$$A_n = \theta^{2-n} \partial_{\bar{z}} \left[\frac{\chi_n(z, \bar{z})}{\theta} - \bar{\chi}_n(z, \bar{z}) \right], \quad \bar{A}_n = \theta^{2-n} \partial_z \left[\frac{\chi_n(z, \bar{z})}{\theta} + \bar{\chi}_n(z, \bar{z}) \right]. \quad (4.22)$$

Notably the infinite hierarchy of functions $\chi_n(z, \bar{z})$ has to fulfil *only* one PDE of second order: the equation (4.20). Its integral can be obtained by determining the general solution of the homogeneous PDE (4.12), *which is the same for all orders* $n \geq 2$ and *one* special solution of the inhomogeneous equation (4.20). It is given by using a Green function that can be chosen to be the same for all orders n without any loss of generality. The solutions of (4.12) can be obtained by employing Bäcklund transformations.

5 Bäcklund transformations

Bäcklund transformations (BTs) are maps between the tangent bundles of integral submanifolds associated with PDEs. If we are able to find a BT from the PDE which we wish to solve and to another one the general integral of which is known, we can obtain the general solution of the first one by integrating the BT. This treatment of a single PDE can be generalized to systems of partial differential equations [9].

By applying BTs we want to reduce the linear PDEs of second order with *nonconstant* coefficients of type (4.12) to linear PDEs with *constant* coefficients. The inhomogeneous extension of equation (4.12) could be solved by BTs, too, but for sake of simplicity we

construct a special solution of the inhomogeneous equation (4.12) by using a Green function and Fourier transformation. Like other authors, e.g. [7], the general form of a BT we are starting from is given by two functions F_1, F_2 :

$$\begin{aligned}\partial_z \hat{\theta}(z, \bar{z}) &= F_1 \left[z, \bar{z}, \theta(z, \bar{z}), \hat{\theta}(z, \bar{z}), \partial_z \theta(z, \bar{z}) \right], \\ \partial_{\bar{z}} \hat{\theta}(z, \bar{z}) &= F_2 \left[z, \bar{z}, \theta(z, \bar{z}), \hat{\theta}(z, \bar{z}), \partial_{\bar{z}} \theta(z, \bar{z}) \right].\end{aligned}\quad (5.1)$$

θ has to fulfil the relation (4.12), whereas $\hat{\theta}$ denotes the transformed function which is supposed to obey a linear PDE with a constant coefficient m^2 : $\partial_z \partial_{\bar{z}} \hat{\theta} - m^2 \hat{\theta} = 0$. Of course m^2 can be equal to zero which would yield the wave equation. Obviously F_1 and F_2 have to fulfil the integrability condition $\partial_z \partial_{\bar{z}} \hat{\theta} = \partial_{\bar{z}} \partial_z \hat{\theta} \Rightarrow dF_1/d\bar{z} = dF_2/dz$. This integrability condition does not lead to a restriction on the solutions of the PDE (4.12), if such a BT is found. Thus F_1 and F_2 have to fulfil the two equations

$$dF_1/d\bar{z} = dF_2/dz, \quad dF_1/d\bar{z} = m^2 \hat{\theta}. \quad (5.2)$$

We would like to point out that the more general ansatz $\partial_z \partial_{\bar{z}} \hat{\theta} = K(z, \bar{z}) \hat{\theta}$ leads to the same BTs (5.9) below at least for the models discussed here [18].

Differentiating the eqs. (5.2) with respect to $\partial_z \theta$ and $\partial_{\bar{z}} \theta$ leads to

$$\partial_{\theta}^2 F_1 = 0, \quad \Rightarrow \quad F_1 = f_1(\theta, \hat{\theta}, z, \bar{z}) \partial_z \theta + m_1(\theta, \hat{\theta}, z, \bar{z}), \quad (5.3)$$

$$\partial_{\hat{\theta}}^2 F_2 = 0, \quad \Rightarrow \quad F_2 = f_2(\theta, \hat{\theta}, z, \bar{z}) \partial_{\bar{z}} \theta + m_2(\theta, \hat{\theta}, z, \bar{z}). \quad (5.4)$$

Substituting these expressions into eqs. (5.2), comparing the coefficients of $\partial_{\bar{z}} \theta$, $\partial_z \theta$ and differentiating with respect to θ , $\hat{\theta}$, we conclude by lengthy but straightforward calculations that

$$\begin{aligned}f_1 &= -f_2 = c_0, \quad c_0 \in \mathcal{R}, \quad c_0 \neq 0, \\ 0 &= \partial_{\theta} m_1 - f_1 \partial_{\hat{\theta}} m_1, \quad \Rightarrow \quad m_1 = g_1(f_1 \theta + \hat{\theta}, z, \bar{z}),\end{aligned}\quad (5.5)$$

$$0 = \partial_{\theta} m_2 + f_1 \partial_{\hat{\theta}} m_2, \quad \Rightarrow \quad m_2 = g_2(f_1 \theta - \hat{\theta}, z, \bar{z}). \quad (5.6)$$

Inserting these results into relations (5.2) and differentiating with respect to $f_1 \theta + \hat{\theta} = \eta_1$ and $f_1 \theta - \hat{\theta} = \eta_2$ we obtain: $\partial_{\eta_1}^2 m_1 = 0$, $\partial_{\eta_2}^2 m_2 = 0$. Substituting $f_1 \hat{\theta} \rightarrow \hat{\theta}$, $f_1 \partial_z \hat{\theta} \rightarrow \partial_z \hat{\theta}$ and $f_1 \partial_{\bar{z}} \hat{\theta} \rightarrow \partial_{\bar{z}} \hat{\theta}$ we get:

$$\partial_z \hat{\theta} = \partial_z \theta + \alpha_1(z, \bar{z}) [\theta + \hat{\theta}] + \beta_1(z, \bar{z}), \quad (5.7)$$

$$\partial_{\bar{z}} \hat{\theta} = -\partial_{\bar{z}} \theta + \alpha_2(z, \bar{z}) [\hat{\theta} - \theta] + \beta_2(z, \bar{z}). \quad (5.8)$$

Considering the two functions β_1, β_2 is only necessary for transformations between *inhomogeneous* PDEs. Thus, we here can choose $\beta_1 = 0 = \beta_2$. Inserting our results (5.7) and (5.8) into equations (5.2) and comparing the coefficients of $\theta, \hat{\theta}$ we finally obtain the linear BT

$$\boxed{\partial_z \hat{\theta} = \partial_z \theta + \{\theta + \hat{\theta}\} \partial_z \psi}, \quad \boxed{\partial_{\bar{z}} \hat{\theta} = -\partial_{\bar{z}} \theta + \{\hat{\theta} - \theta\} \partial_{\bar{z}} \psi}, \quad (5.9)$$

with the BT generating function $\psi = \psi(z, \bar{z})$ which is a *special* solution of

$$\partial_z \partial_{\bar{z}} \psi - (\partial_z \psi)(\partial_{\bar{z}} \psi) - \frac{1}{2} \partial_{\varphi}^2 V(\varphi_0) = 0, \quad \partial_z \partial_{\bar{z}} \psi + (\partial_z \psi)(\partial_{\bar{z}} \psi) - m^2 = 0. \quad (5.10)$$

Substituting $\psi = -\ln(\tilde{\psi})$ in these equations we conclude

$$\partial_z \partial_{\bar{z}} \tilde{\psi} + \frac{1}{2} \partial_\varphi^2 V(\varphi_0) \tilde{\psi} = 0, \quad \partial_z \partial_{\bar{z}} \left(\frac{1}{\tilde{\psi}} \right) - m^2 \left(\frac{1}{\tilde{\psi}} \right) = 0. \quad (5.11)$$

We therefore have to solve the following problem:

2mm We have to find a solution of the equation (4.12) the inverse of which has to fulfil a Klein–Gordon or a wave equation, then we can integrate the linear BT and obtain the general solution of (4.12).

Another method of solving the eqs. (5.10) is based on their linear combinations:

$$\partial_z \partial_{\bar{z}} \psi = \frac{1}{4} (2m^2 + \partial_\varphi^2 V(\varphi_0)) , \quad (\partial_z \psi)(\partial_{\bar{z}} \psi) = \frac{1}{4} (2m^2 - \partial_\varphi^2 V(\varphi_0)) . \quad (5.12)$$

Integrating the first equation and inserting this solution into the second expression imposes a restriction on $\partial_\varphi^2 V(\varphi_0)$. We make use of eqs. (5.12) when we study the ϕ^4 -model.

6 Applications

After solving the homogeneous equations (4.12), we calculate a special solution of their inhomogeneous extension (4.20) by determining a Green function — without need of specifying the inhomogeneity $\widehat{\text{In}h}$. To obtain a solution of equation (4.20) specific for the models under consideration we have to fold the Green function with the inhomogeneity in every order y^n .

The Hamilton–Jacobi theory for the non–selfinteracting scalar field theories $\mathcal{L}_0 = \partial_z \varphi \partial_{\bar{z}} \varphi$ and $\mathcal{L}_1 = \partial_z \varphi \partial_{\bar{z}} \varphi - 1/2 m^2 \varphi^2$ with the light cone variables $z = (x+t)/2$ and $\bar{z} = (x-t)/2$ leads to the wave or the Klein–Gordon equation (4.12) without need for a Bäcklund transformation or specifying an extremal $\varphi_0(z, \bar{z})$:

$$\mathcal{L}_0 : \partial_z \partial_{\bar{z}} \chi_n = \widehat{\text{In}h}(\chi_0, \dots, \chi_{n-1}) , \quad \mathcal{L}_1 : \partial_z \partial_{\bar{z}} \chi_n + m^2 \chi_n = \widehat{\text{In}h}(\chi_0, \dots, \chi_{n-1}) . \quad (6.1)$$

The general solutions of these relations are known. Therefore we draw our attention to the more interesting case of selfinteracting theories:

6.1 The homogeneous equations

6.1.1 Liouville model

Applying our formalism to Liouville’s theory $\mathcal{L} = \partial_z \varphi \partial_{\bar{z}} \varphi + 2 \exp(\varphi)$, using an arbitrary solution of the equation of motion for which the general expression is known [7]:

$$\varphi_0 = \ln \left\{ 2 \frac{(\partial_z s(z))(\partial_{\bar{z}} \bar{s}(\bar{z}))}{(s + \bar{s})^2} \right\} , \quad (6.2)$$

with arbitrary smooth functions $s(z)$ and $\bar{s}(\bar{z})$, the relation (4.12) yields:

$$\partial_z \partial_{\bar{z}} \theta^L - 2 \frac{(\partial_z s)(\partial_{\bar{z}} \bar{s})}{(s + \bar{s})^2} \theta^L = 0, \quad \Rightarrow \quad \partial_s \partial_{\bar{s}} \theta^L(s, \bar{s}) - 2 \frac{1}{(s + \bar{s})^2} \theta^L(s, \bar{s}) = 0 \quad (6.3)$$

by employing a transformation of variables $z \rightarrow s(z)$, $\bar{z} \rightarrow \bar{s}(\bar{z})$. Obviously *one* special solution of this equation is $\theta_0^L = 1/(s+\bar{s})$. Its inverse fulfils the wave equation $\partial_s \partial_{\bar{s}} \hat{\theta}^L = 0$. Thus we know that there exists at least one BT which connects the integral submanifolds of (6.3) and of the wave equation. Returning to the equations (5.11) we conclude that $\psi^L = \ln(s+\bar{s})$. So we can determine the BT by integrating the relations (5.9) and obtain the general solution of the equation (6.3):

$$\theta^L(s, \bar{s}) = \partial_s C(s) - \partial_{\bar{s}} \bar{C}(\bar{s}) + \frac{2}{(s + \bar{s})^2} \{ \bar{C}(\bar{s}) - C(s) \} \quad (6.4)$$

with two arbitrary smooth functions $C(s)$ and $\bar{C}(\bar{s})$.

6.1.2 Sine & Sinh–Gordon model

In two dimensional space–time two types of solitons exist: the “bell” with the same asymptotic value at $x = -\infty$ and $x = \infty$ and a “kink” soliton with different asymptotic values. Moreover there exist a topological conserved quantum number associated with the asymptotic behaviour of these solitons. The corresponding conserved current is given by: $J^\mu = \epsilon^{\mu\nu} \partial_\nu \varphi$ with the antisymmetric tensor $\epsilon^{\mu\nu} = -\epsilon^{\nu\mu}$, $\mu, \nu = 0, 1$. Thus the charge associated with this current is: $N = \int_{-\infty}^{\infty} J^0 dx = \varphi|_{x=\infty} - \varphi|_{x=-\infty}$, which vanishes obviously in the case of the bell solitons. For kinks it is a non–trivial quantum number.

The Sine–Gordon theory possesses a infinite hierarchy of multikink solutions, which can be constructed by Auto–Bäcklund transformations. In this model the conserved quantity is associated with a particle number. For details as to solitons see e.g. [13], [14], [15] and [16].

We want to embed for the Sine–Gordon model $\mathcal{L}_2 = \partial_z \varphi \partial_{\bar{z}} \varphi + 2[1 - \cos(\varphi)]$ the (anti-) kink solution $\varphi_0 = \pm 4 \arctan[\exp(z + \bar{z})]$ and in the case of the Sinh–Gordon model $\mathcal{L}_3 = \partial_z \varphi \partial_{\bar{z}} \varphi - 2[1 - \cosh(\varphi)]$ the bell solution $\varphi_0 = \pm 4 \operatorname{arctanh}[\exp(z + \bar{z})]$, which is only defined for $z + \bar{z} < 0$. We then obtain for equation (4.12):

$$\text{Sine–Gordon:} \quad \partial_z \partial_{\bar{z}} \theta^{SG} - \{2 \tanh^2(z + \bar{z}) - 1\} \theta^{SG} = 0, \quad (6.5)$$

$$\text{Sinh–Gordon:} \quad \partial_z \partial_{\bar{z}} \theta^{Sh} - \{2 \coth^2(z + \bar{z}) - 1\} \theta^{Sh} = 0, \quad (6.6)$$

respectively. Following the discussion of Liouville’s theory we are able to solve these two equations by one BT. For this it is sufficient to realize that the inverse of the two solutions $\theta_0^{SG} = 1/\cosh(z + \bar{z})$ and $\theta_0^{Sh} = 1/\sinh(z + \bar{z})$ solve the Klein–Gordon equation $\partial_z \partial_{\bar{z}} \hat{\theta} = \hat{\theta}$. The functions $\theta_0^{SG} = \tilde{\psi}^{SG}$ and $\theta_0^{Sh} = \tilde{\psi}^{Sh}$ can be calculated by using the relations $\tilde{\psi} = \exp(-\psi)$ and (A.21), (A.2) shown in the Appendix. Thus the two generating functions are $\psi^{SG} = \ln(\cosh(z + \bar{z}))$ and $\psi^{Sh} = \ln(\sinh(z + \bar{z}))$ which determine the BTs (5.9) between the Klein–Gordon eq. and the eqs. (6.5), (6.6). Their general solutions can be calculated by integration of the linear BTs (5.9):

$$\begin{aligned} \theta^{SG} = & \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \exp[-i(qz + \bar{q}\bar{z})] \delta(q\bar{q} + 1) \Upsilon^{SG}(q, \bar{q}) \left\{ 1 - \frac{2}{q^2 + 1} - \frac{2iq}{q^2 + 1} \tanh(z + \bar{z}) \right\} dq d\bar{q} \\ & + c_0 \cosh^{-1}(z + \bar{z}), \end{aligned} \quad (6.7)$$

$$\begin{aligned} \theta^{Sh} = & \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \exp[-i(kz + \bar{k}\bar{z})] \delta(k\bar{k} + 1) \Upsilon^{Sh}(k, \bar{k}) \left\{ \frac{i(k + \bar{k})}{2} - \coth(z + \bar{z}) \right\} dk d\bar{k} \\ & + c_1 \sinh^{-1}(z + \bar{z}), \end{aligned} \quad (6.8)$$

with arbitrary constants $c_0, c_1 \in \mathcal{R}$ and two arbitrary functions $\Upsilon^{SG}(q, \bar{q}), \Upsilon^{Sh}(k, \bar{k})$ which have to be chosen in such a way that the integrals exist. A property of this BTs is that the solution which was used for the transformation, is multiplied with a constant and added to the modified solution of the Klein–Gordon or the wave equation.

This static kink $\varphi_0 = \pm 4 \arctan[\exp(x)]$ can be transformed by a Lorentz boost into time dependent solutions of the relativistic invariant Euler–Lagrange equation $\varphi_0 = \pm 4 \arctan[\exp(\gamma(x - vt) + \delta)], \gamma^2 = 1/(1 - v^2)$ parametrized by the velocity v and the phase shift δ . We are able to include the embedding of these solutions in our discussion making use of the transformation of variables

$$z \rightarrow w = z\gamma(1 - v) + \delta/2, \quad \bar{z} \rightarrow \bar{w} = \bar{z}\gamma(1 + v) + \delta/2. \quad (6.9)$$

The same holds for the static solitons of the relativistic covariant Sinh–, and ϕ^4 –models.

6.1.3 ϕ^4 –model

Contrary to the three previous models the following ones can only be solved by at least two BTs: the ϕ^4 –theories

$$\text{I)} \quad \mathcal{L} = \partial_z \varphi \partial_{\bar{z}} \varphi - 4\varphi^2 + 2\varphi^4 \quad \text{and} \quad \text{II)} \quad \mathcal{L} = \partial_z \varphi \partial_{\bar{z}} \varphi + 2\varphi^2 - 2\varphi^4. \quad (6.10)$$

Both theories have soliton solutions. We choose the (anti-) kink $\varphi_0^I = \pm \tanh(z + \bar{z})$ and in case II the bell solution $\varphi_0^{II} = \pm \cosh^{-1}(z + \bar{z})$. Then we get the two relations (4.12) for θ^I and θ^{II} :

$$\text{I)} : \quad \partial_z \partial_{\bar{z}} \theta^I - \{6 \tanh^2(z + \bar{z}) - 2\} \theta^I = 0, \quad (6.11)$$

$$\text{II)} : \quad \partial_z \partial_{\bar{z}} \theta^{II} - \{6 \tanh^2(z + \bar{z}) - 5\} \theta^{II} = 0. \quad (6.12)$$

Except for the special values of some of the constants they are the same PDEs as that of the Sine–Gordon model, but they cannot be solved by *one* BT only (see Appendix). Therefore we employ two BTs for each model: the first BTs leads to two eqs. in which the coefficient in front of $\tanh^2(z + \bar{z})$ is reduced to 2, the same as the one in the Sine–Gordon theory. This allows us to obtain two Klein–Gordon eqs. which differ by the choice of m^2 after a second BT for each model.

Inserting the function $\partial_\varphi^2 V(\varphi_0) = v(z + \bar{z})$ in the results (A.29) and (A.2) of the Appendix we are able to calculate the functions ψ^I, ψ^{II} and the two solutions of the equations (6.11), (6.12) which are necessary for the transformations:

$$\text{I): } \theta_0^I = \cosh^{-2}(z + \bar{z}), \quad \text{II) } \theta_0^{II} = \exp[d_0(z - \bar{z})] \cosh^{-2}(z + \bar{z}), \quad d_0^2 = 3. \quad (6.13)$$

Thus, with $\psi^I = 2 \ln\{\cosh(z + \bar{z})\}$ and $\psi^{II} = d_0(\bar{z} - z) + 2 \ln\{\cosh(z + \bar{z})\}$ we obtain after one BT and denoting the function $\hat{\theta}$ of eq. (5.9) by $\tilde{\theta}$:

$$\text{I)} : \quad \partial_z \partial_{\bar{z}} \tilde{\theta}^I - \{2 \tanh^2(z + \bar{z}) + 2\} \tilde{\theta}^I = 0, \quad (6.14)$$

$$\text{II)} : \quad \partial_z \partial_{\bar{z}} \tilde{\theta}^{II} - \{2 \tanh^2(z + \bar{z}) - 1\} \tilde{\theta}^{II} = 0. \quad (6.15)$$

The second relation is the same as in the Sine–Gordon theory. Thus we only have to treat the first case here. This PDE has the special solution $\tilde{\theta}_0^I = 1/\cosh(z+\bar{z}) \exp[id_1(z-\bar{z})]$, $d_1^2 = 3$ the inverse of which fulfils a Klein–Gordon equation, namely: $\partial_z \partial_{\bar{z}}(1/\tilde{\theta}_0^I) = 4/\tilde{\theta}_0^I$. The solution $\tilde{\theta}_0^I$ was found according to the method discussed in the Appendix. The generating function for this BT is $\psi_2^I = \ln\{\cosh(z+\bar{z})\} + id_1(\bar{z}-z) = -\ln(\tilde{\theta}_0^I)$. Hence the resulting solutions of the eqs. (6.11) and (6.12) are:

$$\begin{aligned} \theta^I &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \Upsilon^I(q, \bar{q}) \delta(q\bar{q} + 4) \exp[-i(qz + \bar{q}\bar{z})] \left\{ q^4 + 16 - 4q^2 \right. \\ &\quad \left. - (6iq^3 - 24iq) \tanh(z + \bar{z}) - 12q^2 \tanh^2(z + \bar{z}) \right\} dq d\bar{q} \\ &\quad + c_0 \cosh^{-2}(z + \bar{z}) + c_1 \cosh^{-1}(z + \bar{z}) \tanh(z + \bar{z}) \exp[id_1(z - \bar{z})], \end{aligned} \quad (6.16)$$

$$\begin{aligned} \theta^{II} &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \Upsilon^{II}(k, \bar{k}) \delta(k\bar{k} + 1) \exp[-i(kz + \bar{k}\bar{z})] \left\{ -(k^2 + 1)^2 \right. \\ &\quad \left. - 6ik(1 - k^2) \tanh(z + \bar{z}) + 12k^2 \tanh^2(z + \bar{z}) \right\} dk d\bar{k} \\ &\quad + c_2 \cosh^{-2}(z + \bar{z}) \exp[d_0(z - \bar{z})] + c_3 \cosh^{-1}(z + \bar{z}) \tanh(z + \bar{z}) \end{aligned} \quad (6.17)$$

with $d_0^2 = d_1^2 = 3$; $c_0, c_1, c_2, c_3 \in \mathcal{R}$ and two arbitrary functions $\Upsilon^I(q, \bar{q})$, $\Upsilon^{II}(k, \bar{k})$.

Expressions like (6.5) and (6.11) also occur e.g. in stability investigations or in discussion small fluctuations around the known soliton solutions φ_0 of these theories [13][15]. Carrying out the second variation of the action functional in the case of the Sine–Gordon and the ϕ^4 –theory employing the soliton solutions given above — or equivalently inserting $\varphi = \varphi_0 + \varphi_c$, $|\varphi_c| \ll |\varphi_0|$ into the Euler–Lagrange equation of the Sine–Gordon or the ϕ^4 –theory — yields:

$$\left[-\partial_x^2 + n(n+1) \tanh^2(x) + c_0 \right] u(x) = \omega^2 u(x), \quad (6.18)$$

where $\varphi_c(x, t) = \exp(i\omega t)u(x)$. We have $n = 1$, $c_0 = 2$ for the Sine–Gordon and $n = 2$, $c_0 = -2$ for the ϕ^4 –model. Stability of the soliton solutions requires that all eigenvalues ω^2 of this Schrödinger–like equation should be non–negative, so that small perturbations about φ_0 do not grow exponentially in time. The lowest eigenvalue is $\omega^2 = 0$ and the corresponding solution $\hat{\varphi}_c$ is the translation mode. It must be present, because of the translation invariance of our models under consideration. Remarkably we have used it for both models to construct the Bäcklund transformations: $\hat{\varphi}_c^{SG} = 1/\cosh(x = z + \bar{z})$ and $\hat{\varphi}_c^I = 1/\cosh^2(x)$.

The eqs. (6.5) are (6.11) reduced to eqs. (6.18) if one if one sets $x = z + \bar{z}$, $t = z - \bar{z}$ and $\exp(i\omega t)u(x)$. The eqs. (6.18) are solvable by transforming them into hypergeometric differential equations the solutions of which can be given by *finite* series in powers of $\tanh(x)$ functions [17]:

$$\underline{\text{SG}}: \quad \omega_k^2 = k^2 + 1 \quad : \quad u(x) = \exp(ikx) \{k + i \tanh(x)\}, \quad (6.19)$$

$$\underline{\phi_I^4}: \quad \omega^2 = 3 \quad : \quad u(x) = \tanh(x)/\cosh(x), \quad (6.20)$$

$$\omega_k^2 = k^2 + 4 \quad : \quad u(x) = \exp(ikx) \left\{ -1 - k^2 - 3ki \tanh(x) + 3 \tanh^2(x) \right\} \quad (6.21)$$

These solutions are contained in our more general results (6.7) and (6.16).

6.1.4 A mathematical remark

As discussed in the Appendix our results can be generalized in order to reduce the hierarchy of linear PDEs:

$$\partial_z \partial_{\bar{z}} \theta = \{n(n+1)\eta^2 + a\}\theta, \quad a \in \mathcal{R}, \quad n = 0, 1, 2, \dots \quad (6.22)$$

by n BTs to a Klein–Gordon or a wave equation successively. One BT can raise or lower the coefficient n to $n+1$ or $n-1$. We have to assume that the smooth function $\eta(l)$ fulfils the nonlinear differential equation: $\partial_l \eta = \bar{b}\eta^2 + \bar{c}$ with $\bar{c} \in \mathcal{R}$, $\bar{b} = \pm 1$ and $l = z + \bar{z}$:

$$\frac{\bar{b}}{\bar{c}} > 0 : \quad \eta = \sqrt{|\bar{c}|} \tan \left[\sqrt{|\bar{c}|} (l + l_0) \right], \quad \bar{c} \in \mathcal{R}, \quad l_0 \in \mathcal{C}, \quad (6.23)$$

$$\frac{\bar{b}}{\bar{c}} < 0 : \quad \eta = -\sqrt{|\bar{c}|} \tanh \left[\sqrt{|\bar{c}|} (l + l_0) \right], \quad \bar{c} = 0 : \quad \eta = \frac{1}{l_0 - \bar{b}l}. \quad (6.24)$$

The PDEs (6.22) and (6.24) are solvable without need of specifying the constants a or \bar{c} .

Obviously the PDEs (6.11), (6.12), (6.5) and (6.6) correspond to special choices of a, \bar{b}, \bar{c} and n : $\bar{b} = -1, \bar{c} = 1, a = -1$ and $n = 1$ give the Sinh & the Sine–Gordon models, whereas $n = 2, \bar{b} = -1, \bar{c} = 1$ give the ϕ^4 –theory with $a = -2$ (case I) and $a = -1$ (case II).

6.2 The inhomogeneous equation

A special solution of the inhomogeneous equation (4.20) can be obtained for the Sinh & Sine–Gordon and both cases of the ϕ^4 –theory by employing Fourier transformations. We discuss these theories first and return to the Liouville model later.

6.2.1 Sinh & Sine–Gordon, ϕ^4 equations

We introduce the Green function $G(z, \bar{z}, \hat{z}, \hat{\bar{z}})$ for the inhomogeneous equations (4.20):

$$\partial_z \partial_{\bar{z}} G + \frac{1}{2} \left\{ \partial_\varphi^2 V(\varphi_0) \right\} G = \delta(z - \hat{z}) \delta(\bar{z} - \hat{\bar{z}}). \quad (6.25)$$

If a Green function is found, we can calculate the solutions χ_n of eq. (4.20) to all orders of y^n :

$$\chi_n = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \widetilde{\text{Inh}} \left\{ \chi_0(\hat{z}, \hat{\bar{z}}), \dots, \chi_{n-1}(\hat{z}, \hat{\bar{z}}) \right\} G(z, \bar{z}, \hat{z}, \hat{\bar{z}}) d\hat{z} d\hat{\bar{z}}. \quad (6.26)$$

Introducing the Fourier transform \tilde{G} by:

$$G(z, \bar{z}, \hat{z}, \hat{\bar{z}}) = \frac{1}{4\pi^2} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \exp\{i[q(\hat{z} - z) + \bar{q}(\hat{\bar{z}} - \bar{z})]\} \tilde{G}(z, \bar{z}, q, \bar{q}) dq d\bar{q}. \quad (6.27)$$

As we need only a special G we try for \tilde{G} the ansatz that depends on q, \bar{q} and $l = z + \bar{z}$ only. We then obtain

$$1 = \partial_l^2 \tilde{G} - i(q + \bar{q}) \partial_l \tilde{G} - \frac{1}{2} \partial_\varphi^2 V \tilde{G}. \quad (6.28)$$

In our cases the potential term is only a function of l : $\partial_\varphi^2 V = 2a_0 + 2n[n+1]f^2(l)$, $a_0 \in \mathcal{R}$, $n = 1, 2$ and f denotes a $\tanh(l)$ (see (6.5), (6.12) and (6.11)) or $\coth(l)$ (see (6.6)). In order to obtain one solution of the PDE (6.28) it is sufficient to assume any f that solves $\partial_l f = 1 - f^2$. Since (6.28) is only a linear differential equation we are able to calculate a special inhomogeneous solution, if we know a homogeneous one:

$$\tilde{G}_{inh}(q, \bar{q}, l) = \int^l \tilde{G}_{hom}(q, \bar{q}, \hat{l})^{-2} \exp(i[q + \bar{q}]\hat{l}) \int^{\hat{l}} \tilde{G}_{hom}(q, \bar{q}, \tilde{l}) \exp(-i[q + \bar{q}]\tilde{l}) d\tilde{l} d\hat{l}. \quad (6.29)$$

Substituting $\tilde{G}_{hom}(q, \bar{q}, l) = \exp[i(q + \bar{q})l/2] \hat{G}(q, \bar{q}, l)$ leads to:

$$\partial_l^2 \hat{G} - \left\{ a_0 - \frac{(q - \bar{q})^2}{4} + n(n+1)f^2 \right\} \hat{G} = 0. \quad (6.30)$$

Choosing the special ansatz: $\hat{G} = (c_0 + c_1 f) \exp(c_3 l)$ and comparing the coefficients of powers of f we get for $n=1$ with $a = a_0 - (q - \bar{q})^2/4$:

$$a \neq -2 \quad : \quad c_3 = \pm \sqrt{a+2}, \quad c_0 = -c_1 c_3, \quad c_1 \in \mathcal{R}, \quad c_1 \neq 0, \quad (6.31)$$

$$a = -2 \quad : \quad c_3 = 0 = c_0, \quad c_1 \in \mathcal{R}, \quad c_1 \neq 0. \quad (6.32)$$

This choice of parameters provides the Green function for the Sinh & Sine-Gordon theories (6.6) and (6.5). The case $n = 2$ is of interest within the ϕ^4 -models (6.11) and (6.12). Here we have to add the term $c_2 f^2 \exp(c_3 h)$ to \hat{G} . This yields finally:

$$a \neq -6 \quad : \quad 3c_0 = c_2(5 + a), \quad c_2 = -\frac{c_1}{c_3}, \quad c_3 = \pm \sqrt{a+6}, \quad c_1 \in \mathcal{R}, \quad c_1 \neq 0, \quad (6.33)$$

$$a = -6 \quad : \quad c_1 = c_3 = 0, \quad 3c_0 = -2c_2, \quad c_2 \in \mathcal{R}, \quad c_2 \neq 0. \quad (6.34)$$

The sign of c_3 has to be chosen in such a way that the Fourier integral for the Green function $\tilde{G}(z, \bar{z}, q, \bar{q})$ exists.

Inserting these results into the expression for \tilde{G}_{inh} and inverting the Fourier transformation we are able to obtain — in principle — the solution of equation (4.20) for every order of y^n by integration.

6.2.2 Liouville Theorie

For the Liouville model we start with equation (6.25), too and obtain instead of eqs. with relation (6.30), where $l = s(z) + \bar{s}(\bar{z})$:

$$\partial_l^2 \hat{G} - \left\{ -\frac{(q - \bar{q})^2}{4} + \frac{2}{l^2} \right\} \hat{G} = 0. \quad (6.35)$$

The solution is

$$\hat{G} = c_0 \left\{ \frac{1}{l} a \sin(al) - a^2 \cos(al) \right\} + c_1 \left\{ \frac{1}{l} a \cos(al) + a^2 \sin(al) \right\}, \quad a = \pm \frac{(q - \bar{q})}{2}. \quad (6.36)$$

Obviously we can not integrate \tilde{G}_{inh} explicitly, if use of this solution. However, the Green function G can be obtained for the Liouville model. We return to equation (6.25),

choose the ansatz $G(s(z), \hat{s}, \bar{s}(\bar{z}), \hat{\bar{s}}) = H(s(z) - \hat{s})H(\bar{s}(\bar{z}) - \hat{\bar{s}})\bar{G}(s(z), \hat{s}, \bar{s}(\bar{z}), \hat{\bar{s}})$, where H is the usual Heaviside step function. We insert the ansatz into eq. (6.27) and obtain

$$\begin{aligned} 0 = & H(s - \hat{s})H(\bar{s} - \hat{\bar{s}}) \left\{ \partial_s \partial_{\bar{s}} - \frac{2}{(s + \bar{s})^2} \right\} \bar{G} + \delta(s - \hat{s})\delta(\bar{s} - \hat{\bar{s}})[\bar{G} - 1] \\ & + H(s - \hat{s})\delta(\bar{s} - \hat{\bar{s}})\partial_s \bar{G} + H(\bar{s} - \hat{\bar{s}})\delta(s - \hat{s})\partial_{\bar{s}} \bar{G}. \end{aligned} \quad (6.37)$$

In order to solve this equation we have to find one solution \bar{G} of the homogeneous equation (6.3) with the following properties: $\bar{G}(s = \hat{s}, \bar{s} = \hat{\bar{s}}) = 1$, $\partial_{\bar{s}} \bar{G}|_{s=\hat{s}} = 0$ and $\partial_s \bar{G}|_{\bar{s}=\hat{\bar{s}}} = 0$. It can easily be verified that a solution is given by

$$\bar{G} = \frac{1}{\hat{s} + \hat{\bar{s}}} \left\{ 2s(z) - \hat{s} + \hat{\bar{s}} - \frac{2}{s(z) + \bar{s}(\bar{z})} [s(z) + \hat{\bar{s}}][s(z) - \hat{s}] \right\}. \quad (6.38)$$

Thus we have determined an explicit expression for the Green function $G = H(z - \hat{z})H(\bar{z} - \hat{\bar{z}})\bar{G}$.

7 Related extremals

Having determined solutions of the HJE (4.1) combined with the IC (4.2) associated with a given extremal in terms of power series we now want to indicate how new extremals can be generated from a given one.

In order to connect the functions χ_n from equation (4.20), with a one parameter family of extremals $\tilde{\varphi}(z, \bar{z}, u)$ embedded, we expand $\tilde{y} = (\tilde{\varphi} - \varphi_0(z, \bar{z}))$ in the parameter u of the solutions of the equation of motion. Therefore we have to consider the two dimensional submanifold $\Sigma := \{(z, \bar{z}, \tilde{\varphi}(z, \bar{z}))\}$ of the extended configuration space $\mathcal{M}_{2+1} = \{(z, \bar{z}, \varphi)\}$. The starting points of this calculation are the slope functions (4.4):

$$\partial_z \tilde{\varphi}(z, \bar{z}) = \partial_{\varphi} \bar{S}|_{\varphi=\tilde{\varphi}} = \bar{A}_1 + \sum_{i=2}^{\infty} \frac{1}{(i-1)!} \bar{A}_i \tilde{y}^i, \quad \Rightarrow \quad \partial_z \tilde{y} = \sum_{i=1}^{\infty} \frac{1}{i!} \bar{A}_{i+1} \tilde{y}^{i+1}, \quad (7.1)$$

$$\partial_{\bar{z}} \tilde{\varphi}(z, \bar{z}) = \partial_{\varphi} S|_{\varphi=\tilde{\varphi}} = A_1 + \sum_{i=2}^{\infty} \frac{1}{(i-1)!} A_i \tilde{y}^i, \quad \Rightarrow \quad \partial_{\bar{z}} \tilde{y} = \sum_{i=1}^{\infty} \frac{1}{i!} A_{i+1} \tilde{y}^{i+1} \quad (7.2)$$

with $\tilde{y} = \tilde{\varphi} - \varphi_0$, $A_1 = \partial_{\bar{z}} \varphi_0$ and $\bar{A}_1 = \partial_z \varphi_0$. Expanding \tilde{y} in a power series of u

$$\tilde{y}(z, \bar{z}, u) = \tilde{\varphi}(z, \bar{z}, u) - \varphi_0(z, \bar{z}) = u \sum_{k=0}^{\infty} \Lambda_k(z, \bar{z}) \frac{u^k}{k!} \quad (7.3)$$

yields:

$$\partial_{\bar{z}} \sum_{k=0}^{\infty} \frac{1}{k!} \Lambda_k u^{k+1} = \sum_{i=1}^{\infty} \frac{1}{i!} A_{i+1} \left\{ \sum_{j=0}^{\infty} \frac{1}{j!} \Lambda_j u^{j+1} \right\}^i, \quad (7.4)$$

$$\partial_z \sum_{k=0}^{\infty} \frac{1}{k!} \Lambda_k u^{k+1} = \sum_{i=1}^{\infty} \frac{1}{i!} \bar{A}_{i+1} \left\{ \sum_{j=0}^{\infty} \frac{1}{j!} \Lambda_j u^{j+1} \right\}^i. \quad (7.5)$$

Comparing the coefficients in $l+1$ -th order leads to:

$$\partial_{\bar{z}}\Lambda_l = A_2\Lambda_l + \text{Inh}(A_2, \dots, A_{l+1}; \Lambda_0, \dots, \Lambda_{l-1}), \quad (7.6)$$

$$\partial_z\Lambda_l = \bar{A}_2\Lambda_l + \text{Inh}(\bar{A}_2, \dots, \bar{A}_{l+1}; \Lambda_0, \dots, \Lambda_{l-1}), \quad (7.7)$$

with the inhomogeneities Inh which depends on A_i, \bar{A}_i calculated above and the functions of lower order $\Lambda_j, j < l$. Making use of $A_2 = \partial_{\bar{z}} \ln(\theta)$ and $\bar{A}_2 = \partial_z \ln(\theta)$ we get:

$$\begin{aligned} \Lambda_l &= \theta \left\{ \int^z \frac{1}{\theta} \text{Inh}(\bar{A}_2, \dots, \bar{A}_{l+1}; \Lambda_0, \dots, \Lambda_{l-1}) d\hat{z} + \text{const} \right\} \\ &= \theta \left\{ \int^{\bar{z}} \frac{1}{\theta} \text{Inh}(A_2, \dots, A_{l+1}; \Lambda_0, \dots, \Lambda_{l-1}) d\hat{\bar{z}} + \text{const} \right\}. \end{aligned} \quad (7.8)$$

So the solutions of the Euler–Lagrange equations $\tilde{\varphi} = \varphi_0 + \sum_{k=0}^{\infty} \Lambda_k u^{k+1}/k!$ can be obtained by successive integration of the coefficients Λ_l . Regarding the lowest order ($l=0$) $\partial_z\Lambda_0 = \bar{A}_2\Lambda_0, \partial_{\bar{z}}\Lambda_0 = A_2\Lambda_0$ leads to:

$$\Lambda_0 = c_0\theta, \quad c_0 \in \mathcal{R}, \quad \Rightarrow \quad \tilde{y} = uc_0\theta + \dots. \quad (7.9)$$

Since θ obeys the linear PDE (4.12) c_0 can always be absorbed into it. θ is discussed in the ch. Applications above see (6.7), (6.8), (6.16) and (6.17).

Thus the one parametric family of extremals in the vicinity ($u \ll 1$) of the

original solution of the equation of motion $\varphi_0(z, \bar{z})$ is determined by:

$$\tilde{\varphi}(z, \bar{z}, u) = \varphi_0(z, \bar{z}) + u\theta(z, \bar{z}). \quad (7.10)$$

We compare our considerations with an expansion of the field variable $\tilde{\varphi}$ within the Euler–Lagrange equation. $\tilde{\varphi}$ is expanded in the parameter of the families of the extremals denoted as u :

$$\tilde{\varphi}(z, \bar{z}, u) = \tilde{\varphi}_0(z, \bar{z}) + y(z, \bar{z}, u) = \tilde{\varphi}_0(z, \bar{z}) + u \sum_{n=0}^{\infty} \frac{1}{n!} \Lambda_n(z, \bar{z}) u^n. \quad (7.11)$$

$\tilde{\varphi}_0(z, \bar{z})$ is an arbitrary extremal. The potential is assumed to be a analytic function of $\tilde{\varphi}$:

$$V(\tilde{\varphi}) = \sum_{m=0}^{\infty} \frac{1}{m!} \partial_{\tilde{\varphi}}^m V(\tilde{\varphi})|_{\tilde{\varphi}=\tilde{\varphi}_0} (\tilde{\varphi} - \tilde{\varphi}_0)^m = \sum_{m=0}^{\infty} \frac{1}{m!} \partial_{\tilde{\varphi}}^m V(\tilde{\varphi})|_{\tilde{\varphi}=\tilde{\varphi}_0} u^m \left\{ \sum_{n=0}^{\infty} \frac{1}{n!} \Lambda_n(z, \bar{z}) u^n \right\}^m. \quad (7.12)$$

Similar to the results of chapter 3, where we study the Hamilton–Jacobi theory, we obtain by comparing the coefficients of equal order of u^n :

$$u^0 : \quad \partial_z \partial_{\bar{z}} \tilde{\varphi}_0(z, \bar{z}) + \frac{1}{2} \partial_{\tilde{\varphi}}^2 V(\tilde{\varphi})|_{\tilde{\varphi}=\tilde{\varphi}_0} = 0, \quad (7.13)$$

which is fulfilled due to assumption and:

$$u^1 : \quad \partial_z \partial_{\bar{z}} \Lambda_0 + \frac{1}{2} \partial_{\tilde{\varphi}}^2 V(\tilde{\varphi}_0) \Lambda_0 = 0, \quad (7.14)$$

$$u^{n+1} : \quad \partial_z \partial_{\bar{z}} \Lambda_n + \frac{1}{2} \partial_{\tilde{\varphi}}^2 V(\tilde{\varphi}_0) \Lambda_n = \text{Inh}(\Lambda_0 \dots \Lambda_{n-1}, \partial_{\tilde{\varphi}}^2 V(\tilde{\varphi}_0) \dots \partial_{\tilde{\varphi}}^{n+1} V(\tilde{\varphi}_0)). \quad (7.15)$$

The first one of these PDEs yields:

$$\tilde{\varphi} = \tilde{\varphi}_0 + u\Lambda_0. \quad (7.16)$$

Because Λ_0 has to fulfil the same linear equation as θ , which we introduced in the Hamilton–Jacobi theory (4.12), we are able to identify Λ_0 and θ with each other. Thus our result (7.10) is *equivalent* to a second variation of the action functional, which is commonly employed e.g. with semiclassical considerations [13] and stability investigations [15].

The PDE (7.15) is analogous to the equation (4.20), which we obtained in the Hamilton–Jacobi framework. Both can be used to determine the fluctuations in a neighbourhood of a given extremal $\tilde{\varphi}_0$ in every order of u .

If one is able to find the general integral of the Hamilton–Jacobi equation and the integrability condition, the general solution of the Euler–Lagrange equation can be obtained.

7.1 An example

Here we would like to calculate the Hamilton–Jacobi functions $S(z, \bar{z}, \varphi)$, $\bar{S}(z, \bar{z}, \varphi)$ and a related family of extremals $\tilde{\varphi}(z, \bar{z}, u)$ according to the formalism developed in the chapters 4, 6 and 7. For this we choose the Sine–Gordon model with the 1–kink–solution $\varphi_0 = 4 \arctan(\exp(z + \bar{z}))$. Though the functions S , \bar{S} and the related extremals are determined perturbatively the corresponding formal series (4.5), (7.3) can be obtained explicitly.

7.1.1 The Hamilton–Jacobi potentials S , \bar{S}

The coefficients $A_n(z, \bar{z})$ and $\bar{A}_n(z, \bar{z})$ (4.5) are determined by calculating the functions $\chi_n(z, \bar{z})$ and $\bar{\chi}_n(z, \bar{z})$ as solutions of the PDEs (4.20) in every order $n=0, 1, 2, \dots$

The coefficients $A_0(z, \bar{z})$, $\bar{A}_0(z, \bar{z})$ can be calculated from (4.7), (4.8) where the Lagrangian \mathcal{L}_0 is given by

$$\mathcal{L}_0 := \partial_z \varphi_0 \partial_{\bar{z}} \varphi_0 - V(\varphi_0) = 32 \left(\frac{e^{z+\bar{z}}}{1 + e^{2(z+\bar{z})}} \right)^2 = 8 \frac{1}{(\cosh(z + \bar{z}))^2} \quad (7.17)$$

on the single extremal φ_0 . Then we obtain

$$A_0(z, \bar{z}) = -4 \cos \left(\frac{\varphi_0}{2} \right) + \partial_{\bar{z}} \chi_0(z, \bar{z}) = 4 \tanh(z + \bar{z}) + \partial_{\bar{z}} \chi_0(z, \bar{z}), \quad (7.18)$$

$$\bar{A}_0(z, \bar{z}) = -4 \cos \left(\frac{\varphi_0}{2} \right) - \partial_z \chi_0(z, \bar{z}) = 4 \tanh(z + \bar{z}) - \partial_z \chi_0(z, \bar{z}). \quad (7.19)$$

The coefficients of first order are determined by the embedding conditions:

$$A_1(z, \bar{z}) = \partial_{\bar{z}} \varphi_0 = \frac{2}{\cosh l}, \quad \bar{A}_1(z, \bar{z}) = \partial_z \varphi_0 = \frac{2}{\cosh l}, \quad (7.20)$$

with the substitution $l = z + \bar{z}$. Obviously this expressions for the coefficients A_0 , \bar{A}_0 , A_1 , \bar{A}_1 obtained are the general solutions of these equations (7.18), (7.19), (7.20) in contrast

to the following coefficients of higher orders of (4.5).

For the coefficients of the second order $A_2(z, \bar{z})$, $\bar{A}_2(z, \bar{z})$ we choose the translation mode $\theta = 1/\cosh(l)$ of (6.7)

$$A_2(z, \bar{z}) = \partial_{\bar{z}} \ln(\theta) = -\tanh(l), \quad \bar{A}_2(z, \bar{z}) = \partial_z \ln(\theta) = -\tanh(l). \quad (7.21)$$

The inhomogeneity of the equation for $\bar{\chi}_3$ (see (4.17) and (4.22)) always vanishes. Therefore, according to (4.14), (4.17) and the discussion thereby we can choose $\bar{\chi}_3(z, \bar{z}) \equiv 0$ without any loss of generality. Hence the coefficients A_3 and \bar{A}_3 are only determined by the function $\chi_3(z, \bar{z})$, which fulfill the inhomogeneous PDE (4.20):

$$\partial_z \partial_{\bar{z}} \chi_3 - \{2 \tanh^2(z + \bar{z}) - 1\} \chi_3 = -\frac{1}{2} \partial_\varphi^3 V(\varphi)|_{\varphi=\varphi_0} \theta^2 = 2 \frac{\tanh(l)}{\cosh^3(l)}. \quad (7.22)$$

A special solution is given by $\chi_3 = -\tanh(l)/(2 \cosh(l))$, which yields:

$$A_3(z, \bar{z}) = \frac{1}{\theta} \partial_{\bar{z}} \left(\frac{\chi_3}{\theta} \right) = -\frac{1}{2 \cosh(l)}, \quad \bar{A}_3(z, \bar{z}) = \frac{1}{\theta} \partial_z \left(\frac{\chi_3}{\theta} \right) = -\frac{1}{2 \cosh(l)}. \quad (7.23)$$

For $n = 4$ the inhomogeneity of the wave equation (4.14) for the coefficient $\bar{\chi}_4$ (4.17) vanishes. Thus without any loss of generality we may set $\bar{\chi}_4 = 0$. The inhomogeneity of the equation (4.14) for the function χ_4 also vanishes. So we may choose the translation mode again:

$$A_4(z, \bar{z}) = \frac{1}{\theta^2} \partial_{\bar{z}} \left(\frac{\chi_4}{\theta} \right) = \frac{1}{4} \tanh(l), \quad \bar{A}_4(z, \bar{z}) = \frac{1}{\theta^2} \partial_z \left(\frac{\chi_4}{\theta} \right) = \frac{1}{4} \tanh(l). \quad (7.24)$$

Regarding the series of coefficients we see, that those for odd and even indices contain the same functions $1/\cosh(l)$ and $\tanh(l)$, respectively. So we assume the same for the higher orders:

$$A_{2n} = \bar{A}_{2n} = \frac{(-1)^n}{2^{2n-4}} \tanh(l) \quad \text{and} \quad A_{2n+1} = \bar{A}_{2n+1} = \frac{(-1)^n}{2^{2n-3}} \frac{1}{\cosh(l)}, \quad n = 0, 1, 2, 3, \dots \quad (7.25)$$

Inserting these expressions into the formal expansion (4.5) we obtain:

$$\begin{aligned} S(z, \bar{z}, \varphi) &= 4 \left(\sum_{n=0}^{\infty} \frac{(-1)^n y^{2n}}{n! 2^{2n}} \right) \tanh(l) + 4 \left(\sum_{n=0}^{\infty} \frac{(-1)^n y^{2n+1}}{n! 2^{2n+1}} \right) \frac{1}{\cosh(l)} + \partial_{\bar{z}} \chi_0(z, \bar{z}) \\ &= 4 \cos\left(\frac{y}{2}\right) \tanh(l) + 4 \sin\left(\frac{y}{2}\right) \frac{1}{\cosh(l)} + \partial_{\bar{z}} \chi_0(z, \bar{z}) \\ &= -4 \cos\left(\frac{y + \varphi_0}{2}\right) + \partial_z \chi_0(z, \bar{z}) = -4 \cos\left(\frac{\varphi}{2}\right) + \partial_z \chi_0(z, \bar{z}), \end{aligned} \quad (7.26)$$

$$\begin{aligned} \bar{S}(z, \bar{z}, y) &= 4 \left(\sum_{n=0}^{\infty} \frac{(-1)^n y^{2n}}{n! 2^{2n}} \right) \tanh(l) + 4 \left(\sum_{n=0}^{\infty} \frac{(-1)^n y^{2n+1}}{n! 2^{2n+1}} \right) \frac{1}{\cosh(l)} - \partial_{\bar{z}} \chi_0(z, \bar{z}) \\ &= 4 \cos\left(\frac{y}{2}\right) \tanh(l) + 4 \sin\left(\frac{y}{2}\right) \frac{1}{\cosh(l)} - \partial_{\bar{z}} \chi_0(z, \bar{z}) \\ &= -4 \cos\left(\frac{y + \varphi_0}{2}\right) - \partial_{\bar{z}} \chi_0(z, \bar{z}) = -4 \cos\left(\frac{\varphi}{2}\right) - \partial_{\bar{z}} \chi_0(z, \bar{z}). \end{aligned} \quad (7.27)$$

These solutions satisfy the HJE (4.1) and the IC (4.2). The embedding condition (4.4) is fulfilled on the single extremal φ_0 by construction.

7.1.2 Embedded extremals

The embedded extremals $\tilde{\varphi}(z, \bar{z}, u)$ can be determined by a straightforward integration of the eqs.:

$$\partial_z \tilde{\varphi}(z, \bar{z}, u) = \Phi = \partial_\varphi \bar{S}(z, \bar{z}, \varphi = \tilde{\varphi}(z, \bar{z}, u)) = 2 \sin\left(\frac{\tilde{\varphi}}{2}\right), \quad (7.28)$$

$$\partial_{\bar{z}} \tilde{\varphi}(z, \bar{z}, u) = \bar{\Phi} = \partial_\varphi S(z, \bar{z}, \varphi = \tilde{\varphi}(z, \bar{z}, u)) = 2 \sin\left(\frac{\tilde{\varphi}}{2}\right) \quad (7.29)$$

which leads to

$$\int^{\tilde{\varphi}} \frac{dw}{2 \sin(w/2)} = \ln \arctan\left(\frac{\tilde{\varphi}}{4}\right) = z + \bar{f}(\bar{z}), \quad (7.30)$$

$$\int^{\tilde{\varphi}} \frac{dw}{2 \sin(w/2)} = \ln \arctan\left(\frac{\tilde{\varphi}}{4}\right) = \bar{z} + f(z). \quad (7.31)$$

Obviously this system of algebraic equations can only be satisfied by the functions $f(z) = z + u$ and $\bar{f}(\bar{z}) = \bar{z} + u$, $u = \text{const.}$ which gives the family of extremals:

$$\tilde{\varphi}(z, \bar{z}, u) = 4 \arctan(\exp(z + \bar{z} + u)), \quad (7.32)$$

parametrized by one parameter u .

However, here we would like to show how the embedded extremals can be calculated by the recursive formalism developed above, which is necessary if we are not able to determine the DeDonder & Weyl Hamilton–Jacobi functions or the corresponding family of extremals explicitly.

First we calculate the first three orders ($\Lambda_0, \Lambda_1, \Lambda_2$) of the expansion (7.3) of $\tilde{y}(z, \bar{z}, u)$ in u which will turn out to be sufficient to guess the general result for the embedded extremals \tilde{y} . From the eqs. (7.1), (7.2) and (7.3) we get

$$\partial_z \tilde{y} = \bar{A}_2 \tilde{y} + \frac{1}{2!} \bar{A}_3 \tilde{y}^2 + \frac{1}{3!} \bar{A}_4 \tilde{y}^3 + O(u^4), \quad (7.33)$$

$$\partial_{\bar{z}} \tilde{y} = A_2 \tilde{y} + \frac{1}{2!} A_3 \tilde{y}^2 + \frac{1}{3!} A_4 \tilde{y}^3 + O(u^4), \quad (7.34)$$

in which the formal expansion:

$$\tilde{y}(z, \bar{z}, u) = \tilde{\varphi}(z, \bar{z}, u) - \varphi_0(z, \bar{z}) = u \sum_{k=0}^{\infty} \Lambda_k(z, \bar{z}) \frac{u^k}{k!} \quad (7.35)$$

has to be inserted. For the first orders we obtain

$$u^1 : \quad \partial_z \Lambda_0 = \frac{1}{\theta} \partial_z(\theta) \Lambda_0, \quad (7.36)$$

$$\partial_{\bar{z}} \Lambda_0 = \frac{1}{\theta} \partial_{\bar{z}}(\theta) \Lambda_0, \quad (7.37)$$

$$u^2 : \quad \partial_z \Lambda_1 = \frac{1}{\theta} \partial_z(\theta) \Lambda_1 + \frac{1}{2! \theta} \partial_z \left(\frac{\chi_3}{\theta} \right) \Lambda_0^2, \quad (7.38)$$

$$\partial_{\bar{z}} \Lambda_1 = \frac{1}{\theta} \partial_{\bar{z}}(\theta) \Lambda_1 + \frac{1}{2! \theta} \partial_{\bar{z}} \left(\frac{\chi_3}{\theta} \right) \Lambda_0^2, \quad (7.39)$$

$$u^3 : \quad \partial_z \Lambda_2 = 2 \frac{1}{2! \theta} \partial_z(\theta) \Lambda_2 + 2 \frac{1}{3! \theta^2} \partial_z \left(\frac{\chi_4}{\theta} \right) \Lambda_0^3 + 4 \frac{1}{2! \theta} \partial_z \left(\frac{\chi_3}{\theta} \right) \Lambda_0 \Lambda_1, \quad (7.40)$$

$$\partial_{\bar{z}} \Lambda_2 = 2 \frac{1}{2! \theta} \partial_{\bar{z}}(\theta) \Lambda_2 + 2 \frac{1}{3! \theta^2} \partial_{\bar{z}} \left(\frac{\chi_4}{\theta} \right) \Lambda_0^3 + 4 \frac{1}{2! \theta} \partial_{\bar{z}} \left(\frac{\chi_3}{\theta} \right) \Lambda_0 \Lambda_1. \quad (7.41)$$

The solutions of these eqs. are determined up to a constant which may be absorbed by a redefinition of the parameter u .

$$\Lambda_0 = 2\theta = \frac{2}{\cosh(l)}, \quad \Lambda_1 = 2\chi_3 = -\frac{\tanh(l)}{\cosh(l)} = \frac{d}{dl} \left(\frac{1}{\cosh(l)} \right), \quad (7.42)$$

$$\begin{aligned} \Lambda_2 &= \frac{8}{3}\chi_4 + 4\frac{\chi_3^2}{\theta} - \frac{1}{3}\theta = \frac{4}{3\cosh^2(l)} \left(\sinh^2(l) - \frac{1}{2}\cosh^2(l) \right) \\ &= \frac{d^2}{dl^2} \left(\frac{2}{3\cosh(l)} \right). \end{aligned} \quad (7.43)$$

From these coefficients we can already guess the form of the coefficients of the higher orders:

$$\Lambda_k = \frac{1}{k+1} \frac{d^k}{dl^k} \left(\frac{2}{\cosh(l)} \right) \quad (7.44)$$

from which we obtain the embedded extremals:

$$\begin{aligned} \tilde{\varphi} &= \varphi_0 + u \sum_{k=0}^{\infty} \frac{u^k}{(k+1)!} \frac{d^k}{dl^k} \left(\frac{2}{\cosh(l)} \right) = \varphi_0 + \int^l \sum_{k=0}^{\infty} \frac{u^{k+1}}{(k+1)!} \frac{d^{k+1}}{dl'^{k+1}} \left(\frac{2}{\cosh(l')} \right) dl' \\ &= \varphi_0 + \int^l \frac{2}{\cosh(l'+u)} dl' - \int^l \frac{2}{\cosh(l')} dl' = 4 \arctan(\exp(l+u)). \end{aligned} \quad (7.45)$$

Obviously we get a one parametric family of extremals satisfying the E.L.-equations, as well as the equations (7.28) and (7.29). It covers a strip of the extended configuration space: $0 < \varphi < 2\pi$, $z, \bar{z} \in \mathcal{R}$. By translations $\varphi \rightarrow \varphi + 2\pi$ the whole \mathcal{R}^3 parametrized by $z, \bar{z}, \varphi \in \mathcal{R}$ can be covered by families of these extremals with the exception of the parallel planes $\varphi = 2k\pi$, $k = 0, \pm 1, \pm 2, \dots$, which are solutions of the equations of motion, too. These are the so-called “vacuum” solutions in the Sine–Gordon theory. So this set of families of extremals, counted by the integer number k can be completed by these planes $\varphi = 2k\pi$, $k = 0, \pm 1, \pm 2, \dots$, so that the whole space \mathcal{R}^3 is covered by extremals.

8 Wave Fronts

In order to determine the wave fronts we have to turn to Carathéodory’s framework, i.e. it is necessary to transform the DeDonder & Weyl Hamilton–Jacobi functions $S(z, \bar{z}, \varphi)$ and $\bar{S}(z, \bar{z}, \varphi)$ appearing in the expansion (4.5) and by the series (4.22) into those of Carathéodory $S^z(z, \bar{z}, \varphi)$, $S^{\bar{z}}(z, \bar{z}, \varphi)$, namely S^z and $S^{\bar{z}}$ of eq. (3.11).

8.1 Carathéodory’s Hamilton–Jacobi functions

As stated in chapter 3 the two Hamiltonian formulations are algebraically equivalent for fields with only one field component. Thus the Hamiltonian density \mathcal{H} and the momenta p, \bar{p} are the same in both formalisms. Therefore we can use the equality of the momenta in order to determine $S^z, S^{\bar{z}}$ from the functions S and \bar{S} calculated above.

We therefore return to the transversality conditions in (3.9), (3.11) and obtain:

$$\partial_{\bar{z}} S^{\bar{z}} \partial_{\varphi} S^z - \partial_{\bar{z}} S^z \partial_{\varphi} S^{\bar{z}} = p = \partial_{\varphi} S = \sum_{i=0}^{\infty} \frac{1}{i!} A_{i+1} y^i, \quad (8.1)$$

$$\partial_z S^z \partial_{\varphi} S^{\bar{z}} - \partial_z S^{\bar{z}} \partial_{\varphi} S^z = \bar{p} = \partial_{\varphi} \bar{S} = \sum_{i=0}^{\infty} \frac{1}{i!} \bar{A}_{i+1} y^i. \quad (8.2)$$

Inserting the functions S^z and $S^{\bar{z}}$ expanded in powers of $y = \varphi - \varphi_0(z, \bar{z})$, like S and \bar{S} ,

$$S^z(z, \bar{z}, \varphi) = \sum_{i=0}^{\infty} \frac{1}{i!} A_i^z(z, \bar{z}) y^i, \quad S^{\bar{z}}(z, \bar{z}, \varphi) = \sum_{i=0}^{\infty} \frac{1}{i!} A_i^{\bar{z}}(z, \bar{z}) y^i \quad (8.3)$$

and comparing powers of y^n yields:

$$\begin{aligned} \bar{A}_{n+1} - \sum_{i=0}^n \binom{n}{i} \left\{ [\partial_z A_i^z - A_{i+1}^z \partial_z \varphi_0] A_{n-i+1}^{\bar{z}} - [\partial_z A_i^{\bar{z}} - A_{i+1}^{\bar{z}} \partial_z \varphi_0] A_{n-i+1}^z \right\} &= 0, \\ A_{n+1} + \sum_{i=0}^n \binom{n}{i} \left\{ [\partial_{\bar{z}} A_i^z - A_{i+1}^z \partial_{\bar{z}} \varphi_0] A_{n-i+1}^{\bar{z}} - [\partial_{\bar{z}} A_i^{\bar{z}} - A_{i+1}^{\bar{z}} \partial_{\bar{z}} \varphi_0] A_{n-i+1}^z \right\} &= 0. \end{aligned}$$

These equations determine the coefficients $A_{n+1}^z(z, \bar{z})$, $A_{n+1}^{\bar{z}}(z, \bar{z})$ recursively:

$$(\partial_z A_0^z) A_{n+1}^{\bar{z}} - (\partial_z A_0^{\bar{z}}) A_{n+1}^z = \bar{A}_{n+1} + \text{Inh}_1^{n+1}(A_0^z, A_0^{\bar{z}}, \dots, A_n^z, A_n^{\bar{z}}), \quad (8.4)$$

$$(\partial_{\bar{z}} A_0^z) A_{n+1}^{\bar{z}} - (\partial_{\bar{z}} A_0^{\bar{z}}) A_{n+1}^z = A_{n+1} + \text{Inh}_2^{n+1}(A_0^z, A_0^{\bar{z}}, \dots, A_n^z, A_n^{\bar{z}}). \quad (8.5)$$

The functions Inh_1^{n+1} and Inh_2^{n+1} depend only on the coefficients of lower powers of y , namely $A_0^z, A_0^{\bar{z}}, \dots, A_n^z, A_n^{\bar{z}}$. They vanish for $n=0$: $\text{Inh}_1^1 = \text{Inh}_2^1 = 0$. For $n=1$ we get

$$\text{Inh}_1^2 = A_1^{\bar{z}} \partial_z A_1^z - A_1^z \partial_z A_1^{\bar{z}} \quad \text{and} \quad \text{Inh}_2^2 = -A_1^{\bar{z}} \partial_{\bar{z}} A_1^z + A_1^z \partial_{\bar{z}} A_1^{\bar{z}} \quad (8.6)$$

and for arbitrary orders $n > 1$:

$$\begin{aligned} \text{Inh}_1^{n+1} &= - \sum_{i=1}^{n-1} \binom{n}{i} \left\{ [\partial_z A_i^z - A_{i+1}^z \partial_z \varphi_0] A_{n-i+1}^{\bar{z}} - [\partial_z A_i^{\bar{z}} - A_{i+1}^{\bar{z}} \partial_z \varphi_0] A_{n-i+1}^z \right\} \\ &\quad + A_1^{\bar{z}} \partial_z A_n^z - A_1^z \partial_z A_n^{\bar{z}}, \end{aligned} \quad (8.7)$$

$$\begin{aligned} \text{Inh}_2^{n+1} &= + \sum_{i=1}^{n-1} \binom{n}{i} \left\{ [\partial_{\bar{z}} A_i^z - A_{i+1}^z \partial_{\bar{z}} \varphi_0] A_{n-i+1}^{\bar{z}} - [\partial_{\bar{z}} A_i^{\bar{z}} - A_{i+1}^{\bar{z}} \partial_{\bar{z}} \varphi_0] A_{n-i+1}^z \right\} \\ &\quad - A_1^{\bar{z}} \partial_{\bar{z}} A_n^z + A_1^z \partial_{\bar{z}} A_n^{\bar{z}}. \end{aligned} \quad (8.8)$$

The determinant

$$\Delta = (\partial_{\bar{z}} A_0^{\bar{z}})(\partial_z A_0^z) - (\partial_z A_0^{\bar{z}})(\partial_{\bar{z}} A_0^z) = \mathcal{L}|_{\varphi=\varphi_0} = \mathcal{L}_0 = \partial_z \varphi_0 \partial_{\bar{z}} \varphi_0 - V(\varphi_0) \quad (8.9)$$

of this linear system of *algebraic* equations is assumed to be not zero. We consider only those extremals φ_0 and regions in the Minkowski space where the Lagrange density $\mathcal{L}|_{\varphi=\varphi_0}$ does not vanish. It makes sense to exclude those focal points and caustics, where $\mathcal{L}|_{\varphi=\varphi_0} = 0$, because the transversality relations of the wave fronts and extremals are violated otherwise [1].

To obtain the functions $A_0^{\bar{z}}$ and A_0^z we have to consider the zeroth order of the Hamilton–Jacobi equation in (3.11):

$$(\partial_z A_0^{\bar{z}})(\partial_z A_0^z) - (\partial_z A_0^{\bar{z}})(\partial_z A_0^z) = \partial_z \varphi_0 \partial_{\bar{z}} \varphi_0 - V(\varphi_0) = \mathcal{L}_0. \quad (8.10)$$

If $\mathcal{L}_0 \neq 0$ then one of the functions $A_0^z, A_0^{\bar{z}}$ can be chosen arbitrarily and the other one has to be calculated according to this PDE. For example we may choose $A_0^{\bar{z}} = \bar{z}$, $A_0^z = \int \mathcal{L}_0 dz$. In the case of the kink solution $\varphi_0 = \pm 4 \arctan(\exp(z + \bar{z}))$ for the Sine–Gordon theory we get $A_0^z = 8 \tanh(z + \bar{z})$.

The Hamilton–Jacobi equation of Carathéodory is satisfied automatically in any order of y^n , $n \geq 1$, because the corresponding equation of DeDonder & Weyl is fulfilled:

$$\partial_\varphi \{ \partial_z S^z \partial_{\bar{z}} S^{\bar{z}} - \partial_z S^{\bar{z}} \partial_{\bar{z}} S^z + \mathcal{H} \} = \partial_z p + \partial_{\bar{z}} \bar{p} + \partial_\varphi \mathcal{H} = \partial_\varphi \{ \partial_z S + \partial_{\bar{z}} \bar{S} + \mathcal{H} \} = 0. \quad (8.11)$$

The integrability condition holds too, because it imposes a constraint on the slope functions which are independent of the special Hamiltonian description for one component field theories.

The linear system of equations for $A^z(z, \bar{z})$, $A^{\bar{z}}(z, \bar{z})$ can be solved easily:

$$A_{n+1}^z = \frac{1}{\mathcal{L}_0} \left[(\partial_z A_0^z) A_{n+1} + (\partial_z A_0^z) \bar{A}_{n+1} \right] + \widetilde{\text{Inh}}_1^{n+1}(A_0^z, A_0^{\bar{z}}, \dots, A_n^z, A_n^{\bar{z}}), \quad (8.12)$$

$$A_{n+1}^{\bar{z}} = \frac{1}{\mathcal{L}_0} \left[(\partial_z A_0^{\bar{z}}) A_{n+1} + (\partial_z A_0^{\bar{z}}) \bar{A}_{n+1} \right] + \widetilde{\text{Inh}}_2^{n+1}(A_0^z, A_0^{\bar{z}}, \dots, A_n^z, A_n^{\bar{z}}). \quad (8.13)$$

The functions $\widetilde{\text{Inh}}_1^{n+1}$, $\widetilde{\text{Inh}}_2^{n+1}$ are linear combinations of the inhomogeneities (8.7) and (8.8) may easily be determined. So we get e.g. for the coefficients $A_1^z, A_1^{\bar{z}}$ and $A_2^z, A_2^{\bar{z}}$ the expressions:

$$A_1^z = \frac{1}{\mathcal{L}_0} \left[(\partial_z A_0^z) A_1 + (\partial_z A_0^z) \bar{A}_1 \right] = \frac{[(\partial_z A_0^z) \partial_z \varphi_0 + (\partial_z A_0^z) \partial_z \varphi_0]}{\partial_z \varphi_0 \partial_{\bar{z}} \varphi_0 - V(\varphi_0)}, \quad (8.14)$$

$$A_1^{\bar{z}} = \frac{1}{\mathcal{L}_0} \left[(\partial_z A_0^{\bar{z}}) A_1 + (\partial_z A_0^{\bar{z}}) \bar{A}_1 \right] = \frac{[(\partial_z A_0^{\bar{z}}) \partial_z \varphi_0 + (\partial_z A_0^{\bar{z}}) \partial_z \varphi_0]}{\partial_z \varphi_0 \partial_{\bar{z}} \varphi_0 - V(\varphi_0)} \quad (8.15)$$

and

$$A_2^z = \frac{1}{\mathcal{L}_0} \left[(\partial_z A_0^z) (A_2 - A_1^{\bar{z}} \partial_z A_1^z + A_1^z \partial_z A_1^{\bar{z}}) + (\partial_z A_0^z) (\bar{A}_2 + A_1^{\bar{z}} \partial_z A_1^z - A_1^z \partial_z A_1^{\bar{z}}) \right], \quad (8.16)$$

$$A_2^{\bar{z}} = \frac{1}{\mathcal{L}_0} \left[(\partial_z A_0^{\bar{z}}) (A_2 - A_1^{\bar{z}} \partial_z A_1^z + A_1^z \partial_z A_1^{\bar{z}}) + (\partial_z A_0^{\bar{z}}) (\bar{A}_2 + A_1^{\bar{z}} \partial_z A_1^z - A_1^z \partial_z A_1^{\bar{z}}) \right]. \quad (8.17)$$

The two coefficients A_0^z and $A_0^{\bar{z}}$ have to fulfil only one equation (8.10), because the closed 2–form $\Omega = dS^z \wedge dS^{\bar{z}}$ is invariant under transformations $S^z \rightarrow \hat{S}^z(S^z, S^{\bar{z}})$, $S^{\bar{z}} \rightarrow \hat{S}^{\bar{z}}(S^z, S^{\bar{z}})$ with a Jacobi determinant that equals one. A family of wave fronts given by

$$S^z(z, \bar{z}, \varphi) = \sum_{i=0}^{\infty} \frac{1}{i!} A_i^z(z, \bar{z}) (\varphi - \varphi_0(z, \bar{z}))^i = \sigma = \text{const.}, \quad (8.18)$$

$$S^{\bar{z}}(z, \bar{z}, \varphi) = \sum_{i=0}^{\infty} \frac{1}{i!} A_i^{\bar{z}}(z, \bar{z}) (\varphi - \varphi_0(z, \bar{z}))^i = \bar{\sigma} = \text{const.} \quad (8.19)$$

is not changed by this transformation, but only reparametrized [20] $\sigma \rightarrow \hat{\sigma} = \hat{S}^z(\sigma, \bar{\sigma})$, $\bar{\sigma} \rightarrow \hat{\bar{\sigma}} = \hat{S}^{\bar{z}}(\sigma, \bar{\sigma})$.

8.2 An explicit representation for the wave fronts

In order to obtain an explicit expression for the *one dimensional* wave fronts $z(\sigma, \bar{\sigma}, \varphi)$ and $\bar{z}(\sigma, \bar{\sigma}, \varphi)$ — $\sigma, \bar{\sigma}$ fixed — we have to invert the relations (8.19) which can be regarded as the defining equations for these functions $z(\varphi)$, $\bar{z}(\varphi)$. Here they are assumed to be analytic functions of φ :

$$z = \sum_{i=0}^{\infty} \frac{1}{i!} \alpha_i(\sigma, \bar{\sigma}) (\hat{y} = \varphi - \hat{\varphi}_0(\sigma, \bar{\sigma}))^i, \quad \bar{z} = \sum_{i=0}^{\infty} \frac{1}{i!} \bar{\alpha}_i(\sigma, \bar{\sigma}) (\hat{y} = \varphi - \hat{\varphi}_0(\sigma, \bar{\sigma}))^i, \quad (8.20)$$

i.e. they can be expanded in powers of the difference $\hat{y} = \varphi - \hat{\varphi}_0(\sigma, \bar{\sigma})$, where $\hat{\varphi}_0(\sigma, \bar{\sigma}) = \varphi_0(z(\sigma, \bar{\sigma}), \bar{z}(\sigma, \bar{\sigma}))$ denotes the extremal in terms of the variables $\sigma, \bar{\sigma}$. Obviously there is a difference between the quantities $y = \varphi - \varphi_0(z, \bar{z})$ and $\hat{y} = \varphi - \hat{\varphi}_0(\sigma, \bar{\sigma})$. Thus in order to determine the coefficients $\alpha_n, \bar{\alpha}_n$ we have to insert this series into the defining equations for the wave fronts (8.18) and (8.19):

$$\sigma = \sum_{i=0}^{\infty} \frac{y^i}{i!} A_i^z \left(z = \alpha_0(\sigma, \bar{\sigma}) + \sum_{n=1}^{\infty} \frac{\hat{y}^n}{n!} \alpha_n(\sigma, \bar{\sigma}), \bar{z} = \bar{\alpha}_0(\sigma, \bar{\sigma}) + \sum_{m=1}^{\infty} \frac{\hat{y}^m}{m!} \bar{\alpha}_m(\sigma, \bar{\sigma}) \right), \quad (8.21)$$

$$\bar{\sigma} = \sum_{i=0}^{\infty} \frac{y^i}{i!} A_i^{\bar{z}} \left(z = \alpha_0(\sigma, \bar{\sigma}) + \sum_{n=1}^{\infty} \frac{\hat{y}^n}{n!} \alpha_n(\sigma, \bar{\sigma}), \bar{z} = \bar{\alpha}_0(\sigma, \bar{\sigma}) + \sum_{m=1}^{\infty} \frac{\hat{y}^m}{m!} \bar{\alpha}_m(\sigma, \bar{\sigma}) \right) \quad (8.22)$$

where the variables y have to be expanded in powers of \hat{y} , too:

$$y = \varphi - \varphi_0 \left(z = \alpha_0(\sigma, \bar{\sigma}) + \sum_{n=1}^{\infty} \frac{\hat{y}^n}{n!} \alpha_n(\sigma, \bar{\sigma}), \bar{z} = \bar{\alpha}_0(\sigma, \bar{\sigma}) + \sum_{m=1}^{\infty} \frac{\hat{y}^m}{m!} \bar{\alpha}_m(\sigma, \bar{\sigma}) \right). \quad (8.23)$$

The wave fronts on the extremals $\alpha_0(\sigma, \bar{\sigma}), \bar{\alpha}_0(\sigma, \bar{\sigma})$ are determined by the zeroth order of the eqs. (8.21), (8.22) and (8.23):

$$\sigma = A_0^z(z = \alpha_0(\sigma, \bar{\sigma}), \bar{z} = \bar{\alpha}_0(\sigma, \bar{\sigma})) \quad \text{and} \quad \bar{\sigma} = A_0^{\bar{z}}(z = \alpha_0(\sigma, \bar{\sigma}), \bar{z} = \bar{\alpha}_0(\sigma, \bar{\sigma})). \quad (8.24)$$

For the Sine–Gordon theory we obtain from the choice we have made for the functions $A_0^z, A_0^{\bar{z}}$: $\bar{z} = \bar{\alpha}_0(\sigma, \bar{\sigma}) = \sigma$, $z = \alpha_0(\sigma, \bar{\sigma}) = \text{artanh}(\bar{\sigma}) - \sigma$.

Locally the functions $\alpha_0(\sigma, \bar{\sigma})$ and $\bar{\alpha}_0(\sigma, \bar{\sigma})$ are determined uniquely, because the Jacobi determinant $\Delta = (\partial_{\bar{z}} A_0^z)(\partial_z A_0^{\bar{z}}) - (\partial_z A_0^z)(\partial_{\bar{z}} A_0^{\bar{z}}) = \mathcal{L}|_{\varphi=\hat{\varphi}_0}$ of this transformation $z, \bar{z} \rightarrow \sigma, \bar{\sigma}$ does not vanish, as assumed.

Expanding the expressions (8.21) in \hat{y}^n , inserting the series (8.23) and separating the coefficients $\alpha_n(\sigma, \bar{\sigma}), \bar{\alpha}_n(\sigma, \bar{\sigma})$ we obtain a pair of linear algebraic equations for these coefficients, that can be solved like the system (8.4), (8.5). After inserting the coefficients $A_1, \bar{A}_1, \dots, A_n, \bar{A}_n$ of the DeDonder and Weyl Hamilton–Jacobi theory we get the final result for $n \geq 2$:

$$\alpha_n = \frac{(-\mathcal{L}_0)^{n-1}}{\mathcal{H}_0^{n+1}} \left(V(\varphi_0) A_n - (\partial_{\bar{z}} \varphi_0)^2 \bar{A}_n \right) + \text{Inh}(A_1, \bar{A}_1, \dots, A_{n-1}, \bar{A}_{n-1}, \text{deriv.}), \quad (8.25)$$

$$\bar{\alpha}_n = \frac{(-\mathcal{L}_0)^{n-1}}{\mathcal{H}_0^{n+1}} \left(V(\varphi_0) \bar{A}_n - (\partial_z \varphi_0)^2 A_n \right) + \text{Inh}(A_1, \bar{A}_1, \dots, A_{n-1}, \bar{A}_{n-1}, \text{deriv.}). \quad (8.26)$$

The term “deriv.” in the inhomogeneities denotes the derivatives of $A_i^z, A_i^{\bar{z}}, i=1, \dots, n-1$ with respect to z and \bar{z} . The functions $A_n^z, A_n^{\bar{z}}$ and their derivatives are short-cuts of e.g. $\partial_z A_n^z = \partial_z A_n^z|_{z=\alpha_0(\sigma, \bar{\sigma}), \bar{z}=\bar{\alpha}_0(\sigma, \bar{\sigma})}$ and thus depending on $\sigma, \bar{\sigma}$ only.

Obviously a not degenerated transformation $z, \bar{z} \rightarrow \sigma = S^z, \bar{\sigma} = S^{\bar{z}}$ only exists if the Lagrangian \mathcal{L}_0 and the Hamiltonian densities \mathcal{H}_0 on the extremals do not vanish.

The coefficients of zeroth order are given in (8.24), those of the next two powers in \hat{y} are:

$$\alpha_1(\sigma, \bar{\sigma}) = \frac{\partial_{\bar{z}} \varphi_0}{\mathcal{H}_0} = \frac{\partial_{\bar{z}} \varphi_0}{\partial_z \varphi_0 \partial_{\bar{z}} \varphi_0 + V(\varphi_0)}, \quad \bar{\alpha}_1(\sigma, \bar{\sigma}) = \frac{\partial_z \varphi_0}{\mathcal{H}_0} = \frac{\partial_z \varphi_0}{\partial_z \varphi_0 \partial_{\bar{z}} \varphi_0 + V(\varphi_0)}, \quad (8.27)$$

and

$$\begin{aligned} \alpha_2(\sigma, \bar{\sigma}) &= \frac{V(\varphi_0)}{\mathcal{H}_0^3} [-\mathcal{L}_0 \partial_{\bar{z}} \ln(\theta) + \partial_{\bar{z}} (\partial_z \varphi_0 \partial_{\bar{z}} \varphi_0)] \\ &\quad - \frac{\partial_{\bar{z}} \varphi_0 \partial_{\bar{z}} \varphi_0}{\mathcal{H}_0^3} [-\mathcal{L}_0 \partial_z \ln(\theta) + \partial_z (\partial_z \varphi_0 \partial_{\bar{z}} \varphi_0)] + \frac{\partial_{\bar{z}} \varphi_0}{\mathcal{H}_0^2} \partial_{\varphi} V(\varphi_0), \end{aligned} \quad (8.28)$$

$$\begin{aligned} \bar{\alpha}_2(\sigma, \bar{\sigma}) &= \frac{V(\varphi_0)}{\mathcal{H}_0^3} [-\mathcal{L}_0 \partial_z \ln(\theta) + \partial_z (\partial_z \varphi_0 \partial_{\bar{z}} \varphi_0)] \\ &\quad - \frac{\partial_z \varphi_0 \partial_z \varphi_0}{\mathcal{H}_0^3} [-\mathcal{L}_0 \partial_{\bar{z}} \ln(\theta) + \partial_{\bar{z}} (\partial_z \varphi_0 \partial_{\bar{z}} \varphi_0)] + \frac{\partial_z \varphi_0}{\mathcal{H}_0^2} \partial_{\varphi} V(\varphi_0). \end{aligned} \quad (8.29)$$

Thus the wave fronts $z(\sigma, \bar{\sigma}, \varphi)$ and $\bar{z}(\sigma, \bar{\sigma}, \varphi)$ can be calculated from the coefficients A_n, \bar{A}_n of DeDonder and Weyl’s Hamilton–Jacobi framework and the coefficients $A_0^z, A_0^{\bar{z}}$ of Carathéodory’s one. Carathéodory’s coefficients A_n^z and $A_n^{\bar{z}}, n \geq 1$ are not necessarily be involved in the final expressions (8.25), (8.26), since they can be substituted by those of DeDonder and Weyl — determined by Bäcklund transformations.

8.3 An alternative way to determine the wave fronts

If one is not interested to get an explicit representation of Carathéodory’s Hamilton–Jacobi functions $S^z(z, \bar{z}, \varphi), S^{\bar{z}}(z, \bar{z}, \varphi)$ it is possible to get the coefficients $\alpha_n, \bar{\alpha}_n$ of the last paragraph much more easily. However, this method can only be applied, if the Hamiltonian density \mathcal{H} does not vanish! If $\mathcal{H} = 0$ we have to calculate Carathéodory’s Hamilton–Jacobi functions $S^z(z, \bar{z}, \varphi), S^{\bar{z}}(z, \bar{z}, \varphi)$ as described above.

For nonvanishing Hamiltonian densities \mathcal{H} we may change the independent variables z, \bar{z} to $\sigma = S^z(z, \bar{z}, \varphi)$ and $\bar{\sigma} = S^{\bar{z}}(z, \bar{z}, \varphi)$, while the field φ remains unchanged. The functional determinant of this transformation is just \mathcal{H} according to Carathéodory’s Hamilton–Jacobi equation (3.11). Then the wavefronts $z = z(\varphi, \sigma, \bar{\sigma}), \bar{z} = \bar{z}(\varphi, \sigma, \bar{\sigma})$ can be determined explicitly from the equations

$$\partial_{\varphi} z(\varphi, \sigma, \bar{\sigma}) = \frac{p}{\mathcal{H}}, \quad \partial_{\varphi} \bar{z}(\varphi, \sigma, \bar{\sigma}) = \frac{\bar{p}}{\mathcal{H}}. \quad (8.30)$$

These equations are obtained by comparing the coefficients of the wedge products $d\varphi \wedge d\sigma$ and $d\varphi \wedge d\bar{\sigma}$ in the equation (3.11), if the variables $\sigma = S^z$ and $\bar{\sigma} = S^{\bar{z}}$ and φ are regarded as the independent ones [22]. So we can e.g. immediately determine the coefficients of

the first order α_1 and $\bar{\alpha}_1$ (8.27). The remaining coefficients α_n and $\bar{\alpha}_n$ in (8.26) and (8.25) can be determined by expanding the DeDonder and Weyl momenta p , \bar{p} and the Hamiltonian density \mathcal{H} in powers of \hat{y} .

8.4 An example: the Sine–Gordon theory

We return to our example in chapter 7 in order to illustrate the formalism discussed above.

8.4.1 Carathéodory’s Hamilton–Jacobi functions

To obtain Carathéodory’s Hamilton–Jacobi functions $S^z(z, \bar{z}, \varphi)$, $S^{\bar{z}}(z, \bar{z}, \varphi)$ from those of DeDonder & Weyl, (7.26) and (7.27), we first have to solve the equation (8.10) in order to get the coefficients $A_0^z(z, \bar{z})$, $A_0^{\bar{z}}(z, \bar{z})$ of order y^0 . Because in our example the Lagrangian density depends only on $l = z + \bar{z}$ on the extremals it is useful to transform the independent variables z, \bar{z} into $l = z + \bar{z}$ and $\bar{l} = z - \bar{z}$

$$(\partial_l A_0^z)(\partial_l A_0^{\bar{z}}) - (\partial_l A_0^{\bar{z}})(\partial_l A_0^z) = \frac{1}{2} \mathcal{L}_0 = \frac{4}{\cosh^2(l)}. \quad (8.31)$$

Due to the invariance of the 2-form $\Omega = dS^z \wedge dS^{\bar{z}}$ with respect to the transformations $S^z \rightarrow \hat{S}^z(S^z, S^{\bar{z}})$, $S^{\bar{z}} \rightarrow \hat{S}^{\bar{z}}(S^z, S^{\bar{z}})$ with a Jacobi determinant equal to one we can choose any single solution of this equation without any loss of generality, e.g.

$$A_0^z = A_0^z(l) = 4 \tanh(z + \bar{z}), \quad A_0^{\bar{z}} = A_0^{\bar{z}} = -\bar{l} = \bar{z} - z. \quad (8.32)$$

Inserting this result into the equations (8.14) and (8.15) we get the coefficients of first order

$$A_1^z = A_1^z(l) = \frac{2}{\cosh(l)}, \quad A_1^{\bar{z}} = 0, \quad (8.33)$$

which leads, using the eqs. (8.12), (8.13), (8.16) and (8.17), to the coefficients of the second and the third order in y

$$A_2^z = A_2^z(l) = -\tanh(l), \quad A_3^z = A_3^z(l) = -\frac{1}{2 \cosh(l)}, \quad A_2^{\bar{z}} = A_3^{\bar{z}} = 0. \quad (8.34)$$

Inspecting the series of these coefficients we get an ansatz for those of all orders y^n , $n \geq 1$

$$A_{2n}^z = A_{2n}^z(l) = 4 \frac{(-1)^n}{2^{2n}} \tanh(l), \quad A_{2n-1}^z = A_{2n-1}^z(l) = 4 \frac{(-1)^n}{2^{2n-1}} \frac{1}{\cosh(l)}, \quad A_n^{\bar{z}} = 0, \quad (8.35)$$

from which we obtain Carathéodory’s Hamilton–Jacobi functions $S^z(z, \bar{z}, \varphi)$, $S^{\bar{z}}(z, \bar{z}, \varphi)$ by determining the sum (8.3):

$$\begin{aligned} S(z, \bar{z}, \varphi) &= 4 \left(\sum_{n=0}^{\infty} \frac{(-1)^n y^{2n}}{n! 2^{2n}} \right) \tanh(l) + 4 \left(\sum_{n=0}^{\infty} \frac{(-1)^n y^{2n+1}}{n! 2^{2n+1}} \right) \frac{1}{\cosh(l)}, \\ &= 4 \cos\left(\frac{y}{2}\right) \tanh(l) + 4 \sin\left(\frac{y}{2}\right) \frac{1}{\cosh(l)} = -4 \cos\left(\frac{\varphi}{2}\right), \end{aligned} \quad (8.36)$$

$$S^{\bar{z}}(z, \bar{z}, \varphi) = \bar{z} - z. \quad (8.37)$$

These functions satisfy Carathéodory's Hamilton–Jacobi equation (4.1) and the integrability condition (4.2) as well as the embedding conditions (4.3), (4.4). The solutions (8.36), (8.37) for S^z , $S^{\bar{z}}$ are the only ones provided that we have fixed the coefficients A_0^z and $A_0^{\bar{z}}$, because the eqs. (8.12), (8.13), (8.16) and (8.17) lead to unique solutions. The general solution of the system of PDEs (3.11), (4.2) and (4.4) is obtained by applying arbitrary smooth transformations $S^z \rightarrow \Sigma(S^z, S^{\bar{z}})$, $S^{\bar{z}} \rightarrow \bar{\Sigma}(S^z, S^{\bar{z}})$ to the functions (8.36), (8.37) with a Jacobi determinant $\partial_1 \Sigma \partial_2 \bar{\Sigma} - \partial_2 \Sigma \partial_1 \bar{\Sigma} = 1$

$$S^z(z, \bar{z}, \varphi) = \Sigma(\bar{z} - z, -4 \cos(\varphi/2)) \quad \text{and} \quad S^{\bar{z}}(z, \bar{z}, \varphi) = \bar{\Sigma}(\bar{z} - z, -4 \cos(\varphi/2)) . \quad (8.38)$$

8.4.2 An explicit representation of the wave fronts – the singular case

Carathéodory's Hamilton–Jacobi functions S^z , $S^{\bar{z}}$ are given now by the equations (8.38). If we would like to have them in the explicit form $z(\sigma, \bar{\sigma}, \varphi)$, $\bar{z}(\sigma, \bar{\sigma}, \varphi)$, we have to invert the equations (8.18), (8.19). Doing this we get only the zeroth order coefficients

$$\alpha_0(\sigma, \bar{\sigma}) = \frac{1}{2} \left(\operatorname{artanh} \frac{\sigma}{4} + \bar{\sigma} \right) \quad \text{and} \quad \bar{\alpha}_0(\sigma, \bar{\sigma}) = \frac{1}{2} \left(\operatorname{artanh} \frac{\sigma}{4} - \bar{\sigma} \right) \quad (8.39)$$

without any difficulties. All the other coefficients α_n , $\bar{\alpha}_n$, $n \geq 1$ *do not exist*, because the Hamiltonian density $\mathcal{H}_0 = p_0 \bar{p}_0 + V_0(\varphi)$ with $V_0 = 2(\cos(\varphi) - 1)$ vanishes on the given extremals φ_0 (see (8.25) and (8.26))!

From the results (8.36), (8.37) it becomes obvious why this happens. The function (8.37) depends only on the difference $z - \bar{z}$ whereas (8.36) does not depend on the variables z , \bar{z} at all! Therefore it is impossible to invert the equations (8.18), (8.19) to get the functions $z(\sigma, \bar{\sigma}, \varphi)$ and $\bar{z}(\sigma, \bar{\sigma}, \varphi)$ at every point $(\sigma, \bar{\sigma})$ of the parameter space $\Upsilon = \{(\sigma, \bar{\sigma})\}$. Hence the wave fronts are only defined at points $(\sigma, \bar{\sigma}, \varphi)$ for which

$$\varphi \stackrel{!}{=} g(\sigma) = 2 \arccos(\sigma/4) , \quad -4 \leq \sigma \leq 4 \quad (8.40)$$

holds. The real parameter $\bar{\sigma}$ remains arbitrary. Nevertheless the wave fronts are one dimensional straight lines parallel to each other: $z - \bar{z} = \bar{\sigma} = \text{const.}$. They cover the extended configuration space $\mathcal{M}_{2+1} = \{z, \bar{z}, \varphi\}$ as required, but the wave fronts cannot be given as functions $z(\sigma, \bar{\sigma}, \varphi)$ and $\bar{z}(\sigma, \bar{\sigma}, \varphi)$ because they are parallel to the surfaces $\varphi = \text{const.}$.

Due to the transversality condition (3.18) the 2-parametric family of 1-d wave fronts (see Fig. 1)

$$z - \bar{z} = \bar{\sigma} = \text{const.}, \quad \varphi = g(\sigma) = \text{const.} \quad (8.41)$$

and the 1-parametric set of the 2-d extremals $\tilde{\varphi} = 4 \arctan(\exp(l + u))$ intersect each other transversally everywhere in $\mathcal{M}_{2+1} = \{z, \bar{z}, \varphi\}$, because the Lagrangian density \mathcal{L} differs from zero in \mathcal{M}_{2+1} contrary to the Hamiltonian density which vanishes everywhere in \mathcal{M}_{2+1} . The angle $\angle(w, e)$ between the basis vector W of the tangent space $\mathcal{T}_{\mathcal{P}}\mathcal{W}$ of the wave front

$$w = 2 \sin(\varphi/2)(\partial_z + \partial_{\bar{z}}) \quad (8.42)$$

and an arbitrary nonvanishing vector e in the tangent space $\mathcal{T}_P\mathcal{E}$ of the extremal parametrized by the variables $\lambda, \bar{\lambda}$

$$e = \lambda \partial_z + \bar{\lambda} \partial_{\bar{z}} + 2(\lambda + \bar{\lambda}) \sin(\varphi/2) \partial_\varphi, \text{ with } |\lambda| + |\bar{\lambda}| > 0 \quad (8.43)$$

at the point $P = (z, \bar{z}, \varphi)$ is given by

$$\angle(w, e) = \arccos \left(\frac{(w, e)}{|w||e|} \right) = \arccos \left(\frac{\lambda + \bar{\lambda}}{\sqrt{2} \sqrt{\lambda^2 + \bar{\lambda}^2 + 4(\lambda + \bar{\lambda})^2 \sin^2(\varphi/2)}} \right), \quad (8.44)$$

the minimum of which with respect to a variation of the parameters $\lambda, \bar{\lambda}$ gives the angle under which the wavefront intersects the extremal at P :

$$\angle(\mathcal{W}, \mathcal{E}) = \min \angle(w, e) = \arccos \left(\frac{1}{\sqrt{1 + 8 \sin^2(\varphi/2)}} \right) = \arccos \left(\frac{1}{\sqrt{9 - \sigma^2/2}} \right), \quad (8.45)$$

$\angle(\mathcal{W}, \mathcal{E})$ is constant on a given wave front $\sigma = \text{const.}$, $\bar{\sigma} = \text{const.}$. This angle $\angle(\mathcal{W}, \mathcal{E})$ takes its minimum $\angle(\mathcal{W}, \mathcal{E}) = 0$ for the values $\varphi = \varphi_k = 2k\pi$, $k = 0, \pm 1, \pm 2, \dots$, namely $\angle(\mathcal{W}, \mathcal{E}) = 0$. So in the limit $z + \bar{z} = l \rightarrow \pm\infty$ the transversality condition is violated, since the wave fronts and extremals (7.45) are asymptotically parallel at all points P in the planes $\{z, \bar{z} \in \mathcal{R}, \varphi = 2k\pi\} \in \mathcal{M}_{2+1}$ parametrized by the integer number $k = 0, \pm 1, \pm 2, \dots$. Any of these planes separates the two one-parameter families $\tilde{\varphi} = \arctan(\exp(l+u)) + 2(k-1)\pi$ and $\tilde{\varphi} = \arctan(\exp(l+u)) + 2k\pi$. Every extremal of these families “touches” the plane $\varphi = 2k\pi$ in the limits $l \rightarrow -\infty$ or $l \rightarrow \infty$. The maximum value for $\angle(\mathcal{W}, \mathcal{E}) = \arccos(1/3)$ indicates that the wave fronts and the extremals are never perpendicular to each other.

Notice, that the singular situation discussed above is essentially a coordinate singularity resulting from the singular change of variables z, \bar{z} to $\sigma = S^z(z, \bar{z}, \varphi)$ and $\bar{\sigma} = S^{\bar{z}}(z, \bar{z}, \varphi)$. It shows that we cannot choose the parameters $\sigma, \bar{\sigma}, \varphi$ as independent variables to represent the wave fronts. The wave fronts are not degenerated at all.

However the transversality between the wave fronts and extremals is violated in this case on the boundaries of the regions $\mathcal{M}_k = \{2k\pi < \varphi < 2(k+1)\pi, z, \bar{z} \in \mathcal{R}\}$, where the families of extremals (7.45) are defined — a singularity that cannot be circumvented by a coordinate transformation.

These results can be obtained in a straightforward manner by using the DeDonder–Weyl Hamilton–Jacobi functions $S(z, \bar{z}, \varphi)$ and $\bar{S}(z, \bar{z}, \varphi)$ from (7.26), (7.27).

The Hamiltonian density \mathcal{H} vanishes on the extremals $\tilde{\varphi}(z, \bar{z}, u)$ of (7.45):

$$\mathcal{H} = \partial_\varphi S \partial_\varphi \bar{S} + V(\varphi) = (2 \sin(\varphi/2))^2 - 2(1 - \cos(\varphi)) = 0. \quad (8.46)$$

Notice that the “usual” canonical Hamiltonian density:

$$\mathcal{H}_{can.}(x, t) = \frac{1}{2} \pi^2 + \frac{1}{2} (\partial_x \varphi)^2 + (1 - \cos(\varphi)) \quad \text{with} \quad \pi = \frac{\partial \mathcal{L}}{\partial (\partial_t \varphi)} \quad (8.47)$$

does not vanish on the kink solution φ_0 , because the Legendre–Transformation is applied only with respect to the time derivatives $\partial_t \varphi$ of the field φ and does not affect its spatial

derivative $\partial_x \varphi$. This underlines again that the energy density $\mathcal{H}_{can.}$ is different from the “covariant” Hamiltonian density \mathcal{H} (3.5), that we use in this article.

Carathéodory’s Hamilton–Jacobi equation (3.11) on the family of extremals (7.45):

$$\partial_z S^z \partial_{\bar{z}} S^{\bar{z}} - \partial_z S^{\bar{z}} \partial_{\bar{z}} S^z = -\mathcal{H} = 0, \quad (8.48)$$

the solutions of which determine the wave fronts transversal to the extremals we are interested in, shows that the change of the variables z, \bar{z} to $\sigma = S^z(z, \bar{z}, \varphi)$ and $\bar{\sigma} = S^{\bar{z}}(z, \bar{z}, \varphi)$ is *not* a regular one, because \mathcal{H} is nothing but the functional determinant of this transformation.

The general solution of equation (8.48) is:

$$S^z = \Sigma^z(f(z, \bar{z}), \varphi), \quad S^{\bar{z}} = \Sigma^{\bar{z}}(f(z, \bar{z}), \varphi), \quad (8.49)$$

with an arbitrary smooth function $f(z, \bar{z})$. Inserting this result into the conditions on the momenta (3.11):

$$p = 2 \sin(\varphi/2) = \partial_{\bar{z}} S^{\bar{z}} \partial_{\varphi} S^z - \partial_{\bar{z}} S^z \partial_{\varphi} S^{\bar{z}} = +(\partial_z f) (\partial_f \Sigma^z \partial_{\varphi} \Sigma^{\bar{z}} - \partial_{\varphi} \Sigma^z \partial_f \Sigma^{\bar{z}}), \quad (8.50)$$

$$\bar{p} = 2 \sin(\varphi/2) = \partial_z S^z \partial_{\varphi} S^{\bar{z}} - \partial_z S^{\bar{z}} \partial_{\varphi} S^z = -(\partial_{\bar{z}} f) (\partial_f \Sigma^z \partial_{\varphi} \Sigma^{\bar{z}} - \partial_{\varphi} \Sigma^z \partial_f \Sigma^{\bar{z}}) \quad (8.51)$$

shows, after subtracting these PDEs from each other, that the function f depends on the difference $\bar{l} = z - \bar{z}$ only: $f = f(z - \bar{z})$. Due to the invariance of the basic 2-form (3.10) with respect to symplectic transformations we may choose $f = z - \bar{z}$, $\Sigma^z = -4 \cos(\varphi/2)$ and $\Sigma^{\bar{z}} = \bar{z} - z$ without any loss of generality. This result coincides with (8.36) and (8.37).

8.4.3 An explicit representation of the wave fronts – the regular case

If we would like to circumvent the spurious singularity discussed above we have to ensure that the Hamiltonian density does not vanish in \mathcal{M}_{2+1} , e.g. by adding a global constant c_0 to the Lagrangian density:

$$\mathcal{L} \Rightarrow \mathcal{L} = \partial_z \varphi \partial_{\bar{z}} \varphi + 2(1 - \cos(\varphi)) + c_0, \quad (8.52)$$

which does not influence the equation of motion. More generally, we can add any exact two form $\Gamma = d(f(z, \bar{z})dz + \bar{f}(z, \bar{z})d\bar{z})$ to the basic form (3.2), $\Omega \rightarrow \Omega + \Gamma$, without affecting the momenta p, \bar{p} , the slope functions v, \bar{v} and the equation of motion, but modifying the Hamiltonian and the Lagrangian densities \mathcal{H}, \mathcal{L} :

$$\mathcal{H} \rightarrow \mathcal{H} + \partial_{\bar{z}} f - \partial_z \bar{f}, \quad \mathcal{L} \rightarrow \mathcal{L} - \partial_{\bar{z}} f + \partial_z \bar{f}. \quad (8.53)$$

We see that the form of the wave fronts is influenced by adding such a term Γ while the family of extremals (7.45) remains unchanged — a property which is known in mechanics, too. For the sake of simplicity we choose $f=0$, $\bar{f}=c_0 z$ which shifts \mathcal{L} and \mathcal{H} merely by a constant c_0 .

With the exception of the coefficients A_0, \bar{A}_0 of zeroth order, which are of no interest in determining the embedded extremals and the corresponding wave fronts, the DeDonder and Weyl Hamilton–Jacobi functions $S(z, \bar{z}, \varphi)$ and $\bar{S}(z, \bar{z}, \varphi)$ and the resulting momenta $p = \partial_{\varphi} S$, $\bar{p} = \partial_{\varphi} \bar{S}$ are not affected by this shift contrary to the wave fronts.

Using Carathéodory's Hamilton–Jacobi equation (3.11):

$$\partial_z S^z \partial_{\bar{z}} S^{\bar{z}} - \partial_z S^{\bar{z}} \partial_{\bar{z}} S^z = -\mathcal{H} \equiv c_0 \quad (8.54)$$

which represents nothing but the determinant of the linear system of equations (see (3.11))

$$p = 2 \sin(\varphi/2) = \partial_{\bar{z}} S^{\bar{z}} \partial_{\varphi} S^z - \partial_{\bar{z}} S^z \partial_{\varphi} S^{\bar{z}}, \quad (8.55)$$

$$\bar{p} = 2 \sin(\varphi/2) = \partial_z S^z \partial_{\varphi} S^{\bar{z}} - \partial_z S^{\bar{z}} \partial_{\varphi} S^z. \quad (8.56)$$

For the functions $\partial_{\varphi} S^z$ and $\partial_{\varphi} S^{\bar{z}}$ we get a system of decoupled linear PDEs for the functions $S^z, S^{\bar{z}}$

$$(\mathcal{H} \partial_{\varphi} + p \partial_z + \bar{p} \partial_{\bar{z}}) S^z = (-c_0 \partial_{\varphi} + 2 \sin(\varphi/2) [\partial_z + \partial_{\bar{z}}]) S^z = 0, \quad (8.57)$$

$$(\mathcal{H} \partial_{\varphi} + p \partial_z + \bar{p} \partial_{\bar{z}}) S^{\bar{z}} = (-c_0 \partial_{\varphi} + 2 \sin(\varphi/2) [\partial_z + \partial_{\bar{z}}]) S^{\bar{z}} = 0. \quad (8.58)$$

The general solution:

$$S^z = \Sigma(-c_0 z/2 + 2 \cos(\varphi/2), -c_0 \bar{z}/2 + 2 \cos(\varphi/2)), \quad (8.59)$$

$$S^{\bar{z}} = \bar{\Sigma}(-c_0 z/2 + 2 \cos(\varphi/2), -c_0 \bar{z}/2 + 2 \cos(\varphi/2)) \quad (8.60)$$

of this system PDEs is obtained by the method of characteristics. The functions $\Sigma, \bar{\Sigma}$ have only to satisfy Carathéodory's Hamilton–Jacobi equation (3.11). Taking the invariance with respect to symplectic transformations into account we may choose the functions

$$S^z = \sqrt{2} [\bar{z} - (4/c_0) \cos(\varphi/2)] \quad \text{and} \quad S^{\bar{z}} = \sqrt{2} [-c_0 z/2 + 2 \cos(\varphi/2)] \quad (8.61)$$

without any loss of generality.

Obviously the transformation

$$(z, \bar{z}) \rightarrow (\sigma = S^z(z, \bar{z}, \varphi), \bar{\sigma} = S^{\bar{z}}(z, \bar{z}, \varphi)) \quad (8.62)$$

exists now, leading to an explicit representation of the wave fronts:

$$z(\sigma, \bar{\sigma}, \varphi) = -\frac{\sqrt{2}\bar{\sigma}}{c_0} + \frac{4}{c_0} \cos(\varphi/2), \quad \bar{z}(\sigma, \bar{\sigma}, \varphi) = \frac{\sigma}{\sqrt{2}} + \frac{4}{c_0} \cos(\varphi/2) \quad (8.63)$$

with some properties different from those obtained in the “singular” case: the 1-d wave fronts are not straight lines in the extended configuration space. Subtracting the eqs. (8.63) from each other shows that they lie in planes parallel to the l-axis like in the singular case.

Similar to the “singular” case the angle $\angle(w, e)$ between the basis vector of the tangent space $\mathcal{T}_{\mathcal{P}}\mathcal{W}$ of the wave front

$$w = 2 \sin(\varphi/2) (\partial_z + \partial_{\bar{z}}) - c_0 \partial_{\varphi} \quad (8.64)$$

and an arbitrary nonvanishing vector in the tangent space $\mathcal{T}_{\mathcal{P}}\mathcal{E}$ of the extremal

$$e = \lambda \partial_z + \bar{\lambda} \partial_{\bar{z}} + 2(\lambda + \bar{\lambda}) \sin(\varphi/2) \partial_{\varphi}, \quad \text{with } |\lambda| + |\bar{\lambda}| > 0 \quad (8.65)$$

at the point (z, \bar{z}, φ) is given by

$$\angle(w, e) = \arccos \left(\frac{(w, e)}{|w||e|} \right) = \arccos \left(\frac{2(1 - c_0)(\lambda + \bar{\lambda}) \sin(\varphi/2)}{\sqrt{c_0^2 + 8 \sin^2(\varphi/2)} \sqrt{\lambda^2 + \bar{\lambda}^2 + 4(\lambda + \bar{\lambda})^2 \sin^2(\varphi/2)}} \right),$$

the minimum of which with respect to a variation of the parameters $\lambda, \bar{\lambda}$ gives the angle $\angle(\mathcal{W}, \mathcal{E})$ in which the 1-dimensional wavefront intersects the 2-dimensional extremal at P :

$$\angle(\mathcal{W}, \mathcal{E}) = \min \angle(w, x) = \arccos \left(\frac{2\sqrt{2}|1 - c_0| |\sin(\varphi/2)|}{\sqrt{c_0^2 + 8 \sin^2(\varphi/2)} \sqrt{1 + 8 \sin^2(\varphi/2)}} \right). \quad (8.66)$$

Obviously in the case $c_0 = 1$ the wave fronts cross the extremals always *perpendicularly*, because $\angle(\mathcal{W}, \mathcal{E}) = \pi/2$ holds *everywhere* in \mathcal{M}_{2+1} (see Fig. 3).

For $c_0 \neq 0, 1$ the maximum value $\angle(\mathcal{W}, \mathcal{E}) = \pi/2$ is taken only for $\varphi_k = 2k\pi$, $k = 0, \pm 1, \pm 2, \dots$, *contrary* to the singular case $c_0 = 0$, where for these values of the field variable the minimum of the angle $\angle(\mathcal{W}, \mathcal{E})$ vanishes, indicating that the transversality relation between the wave fronts and extremals is violated there. The angle $\angle(\mathcal{W}, \mathcal{E})$ has a local minimum at those points where $|\cos(\angle(\mathcal{W}, \mathcal{E}))|$ becomes maximal:

$$|\cos(\angle(\mathcal{W}, \mathcal{E})_{min1})| = \frac{2\sqrt{2}|1 - c_0|}{3\sqrt{c_0^2 + 8}}, \quad |\cos(\angle(\mathcal{W}, \mathcal{E})_{min2})| = \frac{|1 - c_0|}{1 + |c_0|}. \quad (8.67)$$

We find the first minima $\angle(\mathcal{W}, \mathcal{E})_{min1}$ on planes $\varphi = (2k + 1)\pi$, $k = 0, \pm 1, \pm 2, \dots$ where the cosine of the field variable vanishes. They do exist for all values of the parameter $c_0 \in \mathcal{R} \setminus \{0\}$. For $c_0 \neq -8$ the angle $\angle(\mathcal{W}, \mathcal{E})_{min1}$ differs from zero, i.e. the transversality relation is fulfilled. If $c_0 = -8$ both minima (8.67) coincide, therefore this case is discussed below.

The second minima $\angle(\mathcal{W}, \mathcal{E})_{min2}$ exist only in the range $|c_0| \leq 8$, $c_0 \neq 0$. These minima lie at values of the field variable: $\varphi = 2k\pi \pm 2 \arcsin(\sqrt{|c_0|/8})$, $k = 0, \pm 1, \pm 2, \dots$. If $c_0 < 0$, this angle vanishes, whereas it differs from zero for all $c_0 > 0$.

It results that in the case $c_0 > 0$ or $c_0 < -8$ the wave fronts and extremals are never parallel to each other ($\angle(\mathcal{W}, \mathcal{E})_{min1,2} \neq 0$). This guarantees the transversality relations everywhere in the extended configuration space \mathcal{M}_{2+1} — even on the planes $\varphi_k = 2k\pi$, $k = 0, \pm 1, \pm 2, \dots$ that separate the one parameter families of extremals — contrary to the singular case. This result coincides with the fact that both the Lagrangian and the Hamiltonian densities do not vanish in \mathcal{M}_{2+1} for $c_0 > 0$ or $c_0 < -8$.

For $-8 \leq c_0 < 0$ the minimum $\angle(\mathcal{W}, \mathcal{E})_{min2}$ is equal to zero. This happens just for the points on the plane

$$\varphi = 2k\pi \pm 2 \arcsin(\sqrt{-c_0/8}), \quad k = 0, \pm 1, \pm 2, \dots \quad (8.68)$$

where the shifted Lagrangian density (8.52) vanishes, as expected from the transversality relation (3.18) (see Fig. 2). The case $c_0 = -8$ is a special one: here the two planes (8.68) that exist in every region $2k\pi < \varphi < 2(k+1)\pi$, $z, \bar{z} \in \mathcal{R}$, $k = 0, \pm 1, \pm 2, \dots$ coincide. Hence we get only one plane $\varphi = 2k\pi + \pi$ in the range $2k\pi < \varphi < 2(k+1)\pi$, where the

transversality relations are violated due to a vanishing shifted Lagrangian density.

The dependence of the angle $\angle(\mathcal{W}, \mathcal{E})$ on the constant c_0 shows, that the geometrical properties of the wave fronts and even the transversality relation may be affected by changes of the Lagrangian density that do not influence the equation of motions and the corresponding extremals at all.

Now we would like to show how these results can be obtained using the recursion formulas given in sections (8.1) and (8.2). We first calculate Carathéodory's Hamilton–Jacobi functions $S^z(z, \bar{z}, \varphi)$, $S^{\bar{z}}(z, \bar{z}, \varphi)$ from those of DeDonder & Weyl, (7.26) and (7.27). For the sake of simplicity we choose the shift $c_0 = -2$ in order to eliminate the constant 2 in the Lagrangian of the Sine–Gordon model $\mathcal{L} = \partial_z \varphi \partial_{\bar{z}} \varphi + 2(1 - \cos(\varphi))$.

The coefficients $A_0^z(z, \bar{z})$ and $A_0^{\bar{z}}(z, \bar{z})$ can be chosen as special solutions:

$$A_0^z = A_0^z(l) = 4 \tanh(l) - l, \quad A_0^{\bar{z}} = A_0^{\bar{z}} = -\bar{l} \quad (8.69)$$

of the equation (8.10) written in terms of the variables $l = z + \bar{z}$ and $\bar{l} = z - \bar{z}$.

$$(\partial_l A_0^z)(\partial_l A_0^{\bar{z}}) - (\partial_l A_0^{\bar{z}})(\partial_l A_0^z) = \frac{\mathcal{L}_0}{2} = \frac{4}{\cosh(l)^2} - 1. \quad (8.70)$$

The equations (8.14) and (8.15) yields the coefficients of first order:

$$A_1^z = A_1^z(l) = \frac{2}{\cosh(l)}, \quad A_1^{\bar{z}} = 0, \quad (8.71)$$

and the relations (8.12), (8.13), (8.16) and (8.17) lead to those of the second and the third order in y . They coincide with those of the singular case (8.34) since the ratios $(\partial_z A_0^z)/\mathcal{L}$ and $(\partial_{\bar{z}} A_0^z)/\mathcal{L}$ are the same in the regular and in the singular case. The same holds for the coefficients $A_n^z(z, \bar{z})$ and $A_n^{\bar{z}}(z, \bar{z})$ of higher orders in y . So we get for $n > 0$:

$$A_{2n}^z = A_{2n}^z(l) = 4 \frac{(-1)^n}{2^{2n}} \tanh(l), \quad A_{2n-1}^z = A_{2n-1}^z(l) = 4 \frac{(-1)^n}{2^{2n-1}} \frac{1}{\cosh(l)}, \quad A_n^{\bar{z}} = 0, \quad (8.72)$$

which combine the expressions (8.69) yield the Hamilton–Jacobi functions $S^z(z, \bar{z}, \varphi)$, $S^{\bar{z}}(z, \bar{z}, \varphi)$:

$$\begin{aligned} S(z, \bar{z}, \varphi) &= 4 \left(\sum_{n=0}^{\infty} \frac{(-1)^n y^{2n}}{n! 2^{2n}} \right) \tanh(l) + 4 \left(\sum_{n=0}^{\infty} \frac{(-1)^n y^{2n+1}}{n! 2^{2n+1}} \right) \frac{1}{\cosh(l)} - l, \\ &= 4 \cos\left(\frac{y}{2}\right) \tanh(l) + 4 \sin\left(\frac{y}{2}\right) \frac{1}{\cosh(l)} - l = -4 \cos\left(\frac{\varphi}{2}\right) - z - \bar{z}, \end{aligned} \quad (8.73)$$

$$S^{\bar{z}}(z, \bar{z}, \varphi) = \bar{z} - z. \quad (8.74)$$

Both functions S^z and $S^{\bar{z}}$ satisfy Carathéodory's Hamilton–Jacobi equation (4.1), the integrability criterion (4.2) and the embedding conditions (4.4). They are related to the solutions (8.61) by the transformation

$$S^z \rightarrow \tilde{S}^z = -\frac{1}{\sqrt{2}}(S^z + S^{\bar{z}}), \quad S^{\bar{z}} \rightarrow \tilde{S}^{\bar{z}} = \frac{1}{\sqrt{2}}(S^{\bar{z}} - S^z). \quad (8.75)$$

The wave fronts $z(\sigma, \bar{\sigma}, \varphi)$ and $\bar{z}(\sigma, \bar{\sigma}, \varphi)$ can be determined recursively following the ideas discussed in the section (8.2):

The coefficients $\alpha_0(\sigma, \bar{\sigma}), \bar{\alpha}_0(\sigma, \bar{\sigma})$ of zeroth order in \hat{y} are given by inverting the eqs. (8.24):

$$\sigma = A_0^z(z = \alpha_0(\sigma, \bar{\sigma}), \bar{z} = \bar{\alpha}_0(\sigma, \bar{\sigma})) = 4 \tanh(\alpha_0 + \bar{\alpha}_0) - (\alpha_0 + \bar{\alpha}_0), \quad (8.76)$$

$$\bar{\sigma} = A_0^{\bar{z}}(z = \alpha_0(\sigma, \bar{\sigma}), \bar{z} = \bar{\alpha}_0(\sigma, \bar{\sigma})) = -\alpha_0 + \bar{\alpha}_0, \quad (8.77)$$

which yields:

$$\alpha_0 = \frac{1}{2}(f^{-1}(\sigma) - \bar{\sigma}) \quad \text{and} \quad \bar{\alpha}_0 = \frac{1}{2}(f^{-1}(\sigma) + \bar{\sigma}), \quad (8.78)$$

where the symbol f^{-1} denotes the inverse of the function $f(x) := \tanh(x) - x$. The coefficients of \hat{y}, \hat{y}^2 and \hat{y}^3 are given by (8.26), (8.27), (8.28) and (8.29):

$$\alpha_1(\sigma) = \frac{\partial_z \varphi_0}{\mathcal{H}_0} = \frac{1}{\cosh(f^{-1}(\sigma))}, \quad \bar{\alpha}_1(\sigma) = \frac{\partial_{\bar{z}} \varphi_0}{\mathcal{H}_0} = \frac{1}{\cosh(f^{-1}(\sigma))}, \quad (8.79)$$

$$\alpha_2(\sigma) = -\frac{1}{2} \tanh(f^{-1}(\sigma)), \quad \bar{\alpha}_2(\sigma) = -\frac{1}{2} \tanh(f^{-1}(\sigma)), \quad (8.80)$$

$$\alpha_3(\sigma) = -\frac{1}{4 \cosh(f^{-1}(\sigma))}, \quad \bar{\alpha}_3(\sigma) = -\frac{1}{4 \cosh(f^{-1}(\sigma))}. \quad (8.81)$$

This leads to a general ansatz for the coefficients α_n and $\bar{\alpha}_n$, $n \geq 1$:

$$\alpha_{2n} = \bar{\alpha}_{2n} = 2 \frac{(-1)^n}{2^{2n}} \tanh(f^{-1}(\sigma)), \quad \alpha_{2n-1} = \bar{\alpha}_{2n-1} = 2 \frac{(-1)^n}{2^{2n-1}} \frac{1}{\cosh(f^{-1}(\sigma))}, \quad (8.82)$$

which give the wave fronts

$$\begin{aligned} z(\sigma, \bar{\sigma}, \varphi) &= 2 \left(\sum_{n=0}^{\infty} \frac{(-1)^n y^{2n}}{n! 2^{2n}} \right) \tanh(f^{-1}(\sigma)) + 2 \left(\sum_{n=0}^{\infty} \frac{(-1)^n y^{2n+1}}{n! 2^{2n+1}} \right) \frac{1}{\cosh(f^{-1}(\sigma))} \\ &\quad - 2 \tanh(f^{-1}(\sigma)) + \frac{1}{2}(f^{-1}(\sigma) - \bar{\sigma}) = -2 \cos\left(\frac{\varphi}{2}\right) - \frac{1}{2}(\sigma + \bar{\sigma}), \end{aligned} \quad (8.83)$$

$$\bar{z}(\sigma, \bar{\sigma}, \varphi) = -2 \cos\left(\frac{\varphi}{2}\right) - \frac{1}{2}(\sigma - \bar{\sigma}). \quad (8.84)$$

which are related to the representation (8.63) by the transformation (8.75) applied to $S^z = \sigma$ and $S^{\bar{z}} = \bar{\sigma}$, which reparametrizes the family of wave fronts only.

9 Conclusions

Within the manifest covariant Hamilton–Jacobi canonical frameworks of DeDonder & Weyl and of Carathéodory we have investigated relations between families of extremals and Hamilton–Jacobi wave fronts for 2–dimensional one component field theories. This is of interest, since the dynamics of fields can be described either by the Euler–Lagrange or the Hamilton–Jacobi equations supplemented by the integrability conditions.

We developed a formalism to solve the DeDonder & Weyl Hamilton–Jacobi equation and

the integrability condition perturbatively by expanding the Hamilton–Jacobi functions in powers of the field variable. Starting from a single given extremal it is then possible to calculate new ones from it by using the two DeDonder & Weyl Hamilton–Jacobi functions.

This formalism is useful especially for investigating extremals in the neighbourhood of known extremals in several 2–dimensional field theories: the massless and massive Klein–Gordon, the Sine– and Sinh–Gordon, the Liouville as well as the ϕ^4 theory. In the Sine–, Sinh–Gordon and in the ϕ^4 theory we have studied the embedding of topologically non-trivial soliton solutions. This approach is related to the usual stability investigations of solitons where perturbations are considered which are a product of functions depending on the time and the space variables separately. We determined the *general* solutions of the equations of the second variation by using Bäcklund transformations. In this manner we have obtained all the extremals in the vicinity of a given one.

Calculating proper Hamilton–Jacobi wave fronts makes the use of Carathéodory’s Hamilton–Jacobi functions $S^z(z, \bar{z}, \varphi)$, $S^{\bar{z}}(z, \bar{z}, \varphi)$ necessary. Solving Carathéo’s Hamilton–Jacobi equation is considerably simplified by using the corresponding DeDonder–Weyl Hamilton–Jacobi functions. One first obtains the wave fronts as equipotential surfaces $S^z = \sigma = \text{const.}$, $S^{\bar{z}} = \bar{\sigma} = \text{const.}$ in an implicit form. If the transformation of variables $(z, \bar{z}, \varphi) \rightarrow (\sigma, \bar{\sigma}, \varphi)$ is a regular one, i.e. if Carathéodory’s Hamilton density does not vanish on the family of extremals under consideration, we get an explicit representation $z = z(\varphi, \sigma, \bar{\sigma})$, $\bar{z} = \bar{z}(\varphi, \sigma, \bar{\sigma})$ for the wave fronts.

These general results have been applied in detail to a special single kink solution of the Sine–Gordon equation. After calculating the DeDonder & Weyl Hamilton–Jacobi potentials S , \bar{S} we obtained a corresponding one–parameter family of embedded extremals: a field of kink solutions of constant energy covering the extended configuration space. From the functions S , \bar{S} we have determined Carathéodory’s Hamilton–Jacobi functions S^z , $S^{\bar{z}}$ explicitly. The wave fronts have been determined for the singular ($\mathcal{H}=0$) as well as for the regular case ($\mathcal{H} \neq 0$). In addition the transversality conditions between the wave fronts and the embedded extremals have been analyzed.

A Appendix

Bäcklund transformations are employed to map the integral submanifolds of the PDE (4.12) to those of the Klein–Gordon or wave equation, the *general* solutions of which are known. In this chapter we determine all the functions $\partial_l^2 V(\varphi_0(l))$ with $l = z + \bar{z}$ that allow this map by *one* BT. But unfortunately it turns out that the nonconstant coefficients $\partial_l^2 V(\varphi_0(l))$ in (6.11), (6.12) for the ϕ^4 models do not belong to this class. Thus, we need at least two BTs to connect these two equations to one PDE with constant coefficients. Moreover we have to discuss which relations of type (4.12) can be reduced to free field equations by one or a *finite* number of BTs at all.

We show that there is no BT which relate the PDEs (6.11), (6.12) and $\partial_z \partial_{\bar{z}} \hat{\theta} = m^2 \hat{\theta}$, $m^2 \in \mathcal{R}$. Even if we make the more general assumption that the transformed function $\hat{\theta}$ obeys the eq. $\partial_z \partial_{\bar{z}} \hat{\theta} = m^2(z, \bar{z}) \hat{\theta}$ with a nonconstant but separable coefficient m^2 : $m^2 = B(z) \bar{B}(\bar{z})$ the equations (6.11) and (6.12) for the ϕ^4 theories cannot be reduced by

one BT to such an equation for $\hat{\theta}$ and thus not solved like the PDEs (6.3), (6.5), (6.6) for the Liouville, the Sine- and the Sinh-Gordon models. The reason is that equation $\partial_z \partial_{\bar{z}} \hat{\theta} = m^2(z, \bar{z}) \hat{\theta}$ can be reduced to a Klein-Gordon one by a suitable transformation of variables $z \rightarrow B(z)$ and $\bar{z} \rightarrow \bar{B}(\bar{z})$ [18]. The equations (6.11) and (6.12) can be transformed by *two* BTs to Klein-Gordon equations. This result can be generalized to PDEs of the following type:

$$\partial_z \partial_{\bar{z}} \theta = \{n[n+1]\eta^2(l = z + \bar{z}) + a\}\theta \quad n = 1, 2, \dots, \quad a \in \mathcal{R}, \quad (\text{A.1})$$

if η is a solution of $\partial_l \eta = \bar{b}\eta^2 + \bar{c}$ with $\bar{b}^2 = \pm 1$ and $\bar{c} \in \mathcal{R}$ by h BTs. Especially the PDEs (6.11), (6.12) can be obtained by choosing $\bar{b} = -1$, $\bar{c} = 1$ and $n = 2$. We start from eqs. (5.12), assuming $m^2 \rightarrow m^2(l)$ to be a function of $l = z + \bar{z}$ and rename $\partial_\varphi^2 V(\varphi_0(l))$ as $2v(l)$. From the equations (5.12) we infer

$$\psi = \psi_0 + A(z) + \bar{A}(\bar{z}), \quad \psi_0(l) = \frac{1}{2} \int^l dl' \int^{l'} dl'' (m^2(l'') + v(l'')), \quad (\text{A.2})$$

$$\Rightarrow 0 = \partial_l^2 \psi_0 + (\partial_l \psi_0 + \partial_z A)(\partial_l \psi_0 + \partial_{\bar{z}} \bar{A}) - m^2(l) \quad (\text{A.3})$$

with two arbitrary functions A and \bar{A} . We are interested in the relation between $m^2(l)$ and $v(l)$. First we have to determine the functions A and \bar{A} , which depend only on z and \bar{z} respectively. Differentiating the PDE (A.3) with respect to $\bar{l} = z - \bar{z}$ leads to:

$$\partial_l \psi_0 [\partial_z^2 A - \partial_{\bar{z}}^2 \bar{A}] = \partial_{\bar{z}}^2 \bar{A} \partial_z A - \partial_z^2 A \partial_{\bar{z}} \bar{A}, \quad (\text{A.4})$$

because ψ_0 depends on l only. Now, if $\partial_{\bar{z}}^2 \bar{A} = 0$ or $\partial_z^2 A = 0$, we obtain from eq. (A.4) that either $\partial_{\bar{z}}^2 \bar{A} = 0$ and $\partial_z^2 A = 0$ (case I) or $\partial_l \psi_0 = -a_0$ (case II). The case II leads to $\partial_l^2 \psi_0 = 0$ and using relation (A.3) gives $m^2(l) = 0$. The definition of ψ_0 finally yields $v = 0$, which is of no interest for us. Without any loss of generality we have chosen: $\partial_z^2 A = 0 \Rightarrow \partial_z A = a_0$, $a_0 \in \mathcal{R}$.

If $0 = \partial_z^2 A - \partial_{\bar{z}}^2 \bar{A}$ (case III), then it follows that either $\partial_z^2 A = \partial_{\bar{z}}^2 \bar{A} = 0$ or $\partial_z A = \partial_{\bar{z}} \bar{A} = a = \text{const.}$, the second case being a special case of the first one. So we have $\partial_z A = a_1$ and $\partial_{\bar{z}} \bar{A} = \bar{a}_1$ with $a_1, \bar{a}_1 \in \mathcal{R}$. Thus case III is contained in the first one.

If $\partial_z^2 A - \partial_{\bar{z}}^2 \bar{A} \neq 0$ (case IV) we are able to divide the relation (A.3) by it and to apply the operator ∂_l once more:

$$0 = (\partial_z^3 A \partial_{\bar{z}}^2 \bar{A} + \partial_z^2 A \partial_{\bar{z}}^3 \bar{A})(\partial_z A - \partial_{\bar{z}} \bar{A}) - 2\partial_z^2 A \partial_{\bar{z}}^2 \bar{A}(\partial_z^2 A - \partial_{\bar{z}}^2 \bar{A}), \quad (\text{A.5})$$

$$\Rightarrow 0 = \partial_z \left\{ \frac{\partial_z^3 A}{\partial_z^2 A} \right\} - 2\partial_z \left\{ \frac{\partial_z^2 A - \partial_{\bar{z}}^2 \bar{A}}{\partial_z A - \partial_{\bar{z}} \bar{A}} \right\}. \quad (\text{A.6})$$

Differentiating this equation with respect to \bar{z} , dividing the result by $\partial_{\bar{z}}^2 \bar{A}$ and applying $\partial_{\bar{z}}$ leads to:

$$\partial_{\bar{z}}^3 \bar{A} = \partial_{\bar{z}}^2 \bar{A} [e_0 \partial_{\bar{z}} \bar{A} + e_1], \quad \partial_z^3 A = \partial_z^2 A [e_2 \partial_z A + e_3], \quad e_0, e_1, e_2, e_3 \in \mathcal{R}. \quad (\text{A.7})$$

Substituting these results into the equation (A.5) yields for $A(z)$ and $\bar{A}(\bar{z})$ the differential equations:

$$\partial_{\bar{z}}^2 \bar{A} = \frac{1}{2} (\partial_{\bar{z}} \bar{A})^2 e_0 + e_4 \partial_{\bar{z}} \bar{A} + e_5, \quad \partial_z^2 A = \frac{1}{2} (\partial_z A)^2 e_0 + e_4 \partial_z A + e_5, \quad (\text{A.8})$$

with the same constants e_0, e_4, e_5 for both functions. Inserting these expressions into relation (A.4) we get for $\partial_l^2 \psi_0 = (m^2 + v)/2$:

$$\partial_l^2 \psi_0 = -\frac{e_0 \partial_z^2 A \partial_{\bar{z}}^2 \bar{A}}{2[\frac{e_0}{2}(\partial_z A + \partial_{\bar{z}} \bar{A}) + e_4]^2}. \quad (\text{A.9})$$

From equation (A.3) it follows that

$$m^2(l) = \left(1 - \frac{e_0}{2}\right) \partial_z^2 A \partial_{\bar{z}}^2 \bar{A} \left[\frac{e_0}{2}(\partial_z A + \partial_{\bar{z}} \bar{A}) + e_4\right]^{-2}, \quad (\text{A.10})$$

$$v(l) = -\left(1 + \frac{e_0}{2}\right) \partial_z^2 A \partial_{\bar{z}}^2 \bar{A} \left[\frac{e_0}{2}(\partial_z A + \partial_{\bar{z}} \bar{A}) + e_4\right]^{-2}. \quad (\text{A.11})$$

Thus we have to solve the differential equations (A.7) for $\partial_z A$ and $\partial_{\bar{z}} \bar{A}$ with different choices of the constants e_0, e_4 and e_5 . Depending on the sign of $\Delta = 2e_0 e_5 - e_4^2$ there exist three different cases:

$$\text{IV.1) } \Delta = 0 \quad : \quad \partial_z A = -\frac{e_4}{e_0} - \frac{2}{c(z - z_0)}, \quad \partial_{\bar{z}} \bar{A} = -\frac{e_4}{e_0} - \frac{2}{c(\bar{z} - \bar{z}_0)}, \quad z_0, \bar{z}_0 \in \mathbb{C} \quad (\text{A.12})$$

$$\text{IV.2) } \Delta < 0 \quad : \quad \partial_z A = \frac{\sqrt{\Delta}}{e_0} \tan\left(\frac{\sqrt{\Delta}}{2}(z - z_0)\right) - \frac{e_4}{e_0}, \quad (\text{A.13})$$

$$\partial_{\bar{z}} \bar{A} = \frac{\sqrt{\Delta}}{e_0} \tan\left(\frac{\sqrt{\Delta}}{2}(\bar{z} - \bar{z}_0)\right) - \frac{e_4}{e_0}, \quad (\text{A.14})$$

$$\text{IV.3) } \Delta > 0 \quad : \quad \partial_z A = -\frac{\sqrt{-\Delta}}{e_0} \tanh\left(\frac{\sqrt{-\Delta}}{2}(z - z_0)\right) - \frac{e_4}{e_0}, \quad (\text{A.15})$$

$$\partial_{\bar{z}} \bar{A} = -\frac{\sqrt{-\Delta}}{e_0} \tanh\left(\frac{\sqrt{-\Delta}}{2}(\bar{z} - \bar{z}_0)\right) - \frac{e_4}{e_0}. \quad (\text{A.16})$$

For the case IV.1) we obtain:

$$m^2(l) = \frac{4}{e_0^2} \left(1 - \frac{e_0}{2}\right) [l - l_0]^{-2}, \quad v = -\frac{4}{e_0^2} \left(1 + \frac{e_0}{2}\right) [l - l_0]^{-2}, \quad l_0 = z_0 + \bar{z}_0. \quad (\text{A.17})$$

If we choose $e_0 = 2$, then $m^2(l)$ vanishes and $v = -2[l - l_0]^{-2}$ is for $l_0 = 0$ identical with $\partial_\varphi^2 V(\varphi_0)$ in Liouville's model and on the other hand if one would like to obtain the PDE (6.3) only (this means $m^2(l) = -2[l - l_0]^{-2}$) one has to calculate e_0 from the eq. $-2 = 4(1 - e_0/2)/e_0^2$ yielding $e_{0,1,2} = -1, 2$. Therefore one obtains the wave equation or once again Liouville's model. The last case represents only a map of (6.3) onto itself (for details of Auto-BTs see e.g. [8]).

The case IV.2) yields:

$$m^2(l) = \frac{\Delta}{e_0^2} \left(1 - \frac{e_0}{2}\right) \sin^{-2} \left[\frac{\sqrt{\Delta}}{2}(l - l_0)\right], \quad v = -\frac{\Delta}{e_0^2} \left(1 + \frac{e_0}{2}\right) \sin^{-2} \left[\frac{\sqrt{\Delta}}{2}(l - l_0)\right]. \quad (\text{A.18})$$

This is of no interest for our special models. Notice, however, that the special choice $v = -\Delta 2^{-1} \sin^{-2}(\sqrt{\Delta}[l - l_0]/2)$ leads to the wave equation.

The case IV.3) is obviously similar to IV.2). The resulting functions $m^2(l)$ and $v(l)$ are:

$$m^2(l) = \frac{\Delta}{e_0^2} \left(1 - \frac{e_0}{2}\right) \sinh^{-2} \left[\frac{\sqrt{-\Delta}}{2}(l - l_0)\right], \quad v = \frac{\Delta}{e_0^2} \left(1 + \frac{e_0}{2}\right) \sinh^{-2} \left[\frac{\sqrt{-\Delta}}{2}(l - l_0)\right]. \quad (\text{A.19})$$

As the equations (6.11) and (6.12) for the ϕ^4 models are not contained in the cases discussed up to now we return to case I: $\partial_z A = a_1$ and $\partial_{\bar{z}} \bar{A} = \bar{a}_1$. Assuming m^2 to be independent of l equation (A.3) then gives:

$$\partial_l^2 \psi_0 + (\partial_l \psi_0)^2 + (a_1 + \bar{a}_1) \partial_l \psi_0 + a_1 \bar{a}_1 - m^2 = 0. \quad (\text{A.20})$$

Substituting $\xi = \partial_l \psi_0 + (a_1 + \bar{a}_1)/2$ and $\tilde{\Delta} = m^2 + (a_1 - \bar{a}_1)^2/4$ leads to:

$$\text{I.1)} \quad \tilde{\Delta} > 0 : \xi = \tilde{\Delta} \tanh(\tilde{\Delta}(l - l_0)), \Rightarrow v = 2\tilde{\Delta}^2(1 - \tanh^2[\tilde{\Delta}(l - l_0)]) - m^2 \quad (\text{A.21})$$

$$\text{I.2)} \quad \tilde{\Delta} < 0 : \xi = -\tilde{\Delta} \tan(\tilde{\Delta}(l - l_0)), \Rightarrow v = -2\tilde{\Delta}^2(1 + \tan^2[\tilde{\Delta}(l - l_0)]) - m^2 \quad (\text{A.22})$$

$$\text{I.3)} \quad \tilde{\Delta} = 0 : \xi = \frac{1}{l - l_0}, \Rightarrow v = -2\frac{1}{(l - l_0)^2} - m^2. \quad (\text{A.23})$$

The case I.1 is the essential one for us. Because of the special type of the equations (6.11) and (6.12), we are able to eliminate Δ by a transformation of variables $z \rightarrow \Delta z$ and $\bar{z} \rightarrow \Delta \bar{z}$. Thus only the class of PDEs with a coefficient 2 in front of their $\tanh^2(l)$:

$$\partial_z \partial_{\bar{z}} \theta = \{n_0 \tanh^2(l) + \bar{n}_0\} \theta, \quad n_0 = 2, \quad \bar{n}_0 \in \mathcal{R} \quad (\text{A.24})$$

can be reduced by *one* BT to a Klein–Gordon or wave equation, like the relations we obtain in the Sine–Gordon (6.5) and the Sinh–Gordon (6.6) theory. For the ϕ^4 models with $n_0=6$ we need at least two BTs.

Substituting $m^2(l) = 2\psi_0(l) - v(l)$ in equation (A.3) leads to:

$$\partial_l^2 \psi_0 - (\partial_l \psi_0 + a_1)(\partial_l \psi_0 + \bar{a}_1) - v = 0. \quad (\text{A.25})$$

Choosing v to be equal to $-b\eta^2(l) - c$ where the function η has to obey the equation $\partial_l \eta = \bar{b}\eta^2 + \bar{c}$ with $\bar{b}, \bar{c} \in \mathcal{R}$, making the special ansatz $\partial_l \psi_0 = d_0 + d_1 \eta$ with two constants d_0, d_1 , inserting all this into equation (A.25) and comparing the coefficients in front of the powers of η yields:

$$\underline{\eta^2} : d_1 \bar{b} + b - d_1^2 = 0, \Rightarrow d_{1,2} = \frac{\bar{b}}{2} \pm \sqrt{\left(\frac{\bar{b}}{2}\right)^2 + b}, \quad (\text{A.26})$$

$$\underline{\eta^1} : 2d_0 + a_1 + \bar{a}_1 = 0, \Rightarrow 2d_0 = -a_1 - \bar{a}_1, \quad (\text{A.27})$$

$$\underline{\eta^0} : d_1 \bar{c} + c - (d_0 + a_1)(d_0 + \bar{a}_1) = 0, \Rightarrow \bar{a}_1 = \pm 2\sqrt{-d_1 \bar{c} - c} + a_1. \quad (\text{A.28})$$

For $m^2(l)$ we obtain:

$$m^2(l) = 2\partial_l^2 \psi_0 - v = c + 2d_1 \bar{c} + (2d_1 \bar{b} + b)\eta^2. \quad (\text{A.29})$$

Of special interest is the coefficient in front of η^2 . Inserting d_1 into eq. (A.29) we get:

$$2d_1 \bar{b} + b = \bar{b}^2 + b \pm \bar{b}\sqrt{\bar{b}^2 + 4b}, \quad (\text{A.30})$$

with $b = n(n+1)$ and choosing \bar{b} to be equal to ± 1 we obtain:

$$2d_1 \bar{b} + b = n(n+1) + 1 \pm \sqrt{1 + 4n(n+1)} = \begin{cases} n(n-1) \\ (n+1)(n+2) \end{cases}. \quad (\text{A.31})$$

Thus starting with a coefficient $n(n+1)$ one BT can raise or lower n by 1. Choosing $n = 1, 2, \dots$ we are able to calculate the functions ψ_i , which are essential in order to determine the i -th BT of the hierarchy of n BTs, using the equation (A.2). Moreover the solutions $\tilde{\psi}_i = \exp(-\psi_i)$ of the i -th PDE $\partial_z \partial_{\bar{z}} \theta + v_i \theta = 0$ which reciprocal fulfils $\partial_z \partial_{\bar{z}} \hat{\theta} + m_i^2(l) \hat{\theta} = 0$ (see (5.9)) can be calculated. Obviously the special case $\bar{b} = -1$ and $\bar{c} = 1$ leads to:

$$\partial_z \partial_{\bar{z}} \theta = [n(n+1) \tanh^2(z + \bar{z}) + a] \theta. \quad (\text{A.32})$$

For $n = 2$ and special choices of a this relation yields the PDEs (6.11), (6.12) for the ϕ^4 -theories. So they can be solved by $n=2$ BTs.

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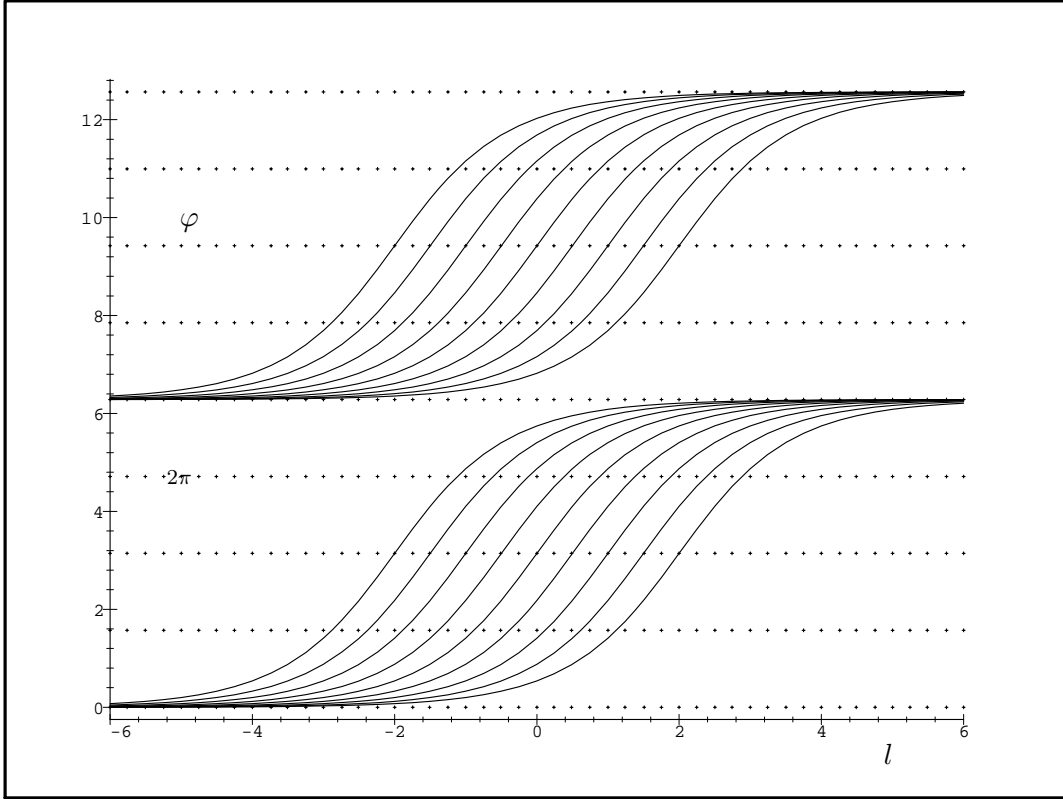
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Figures

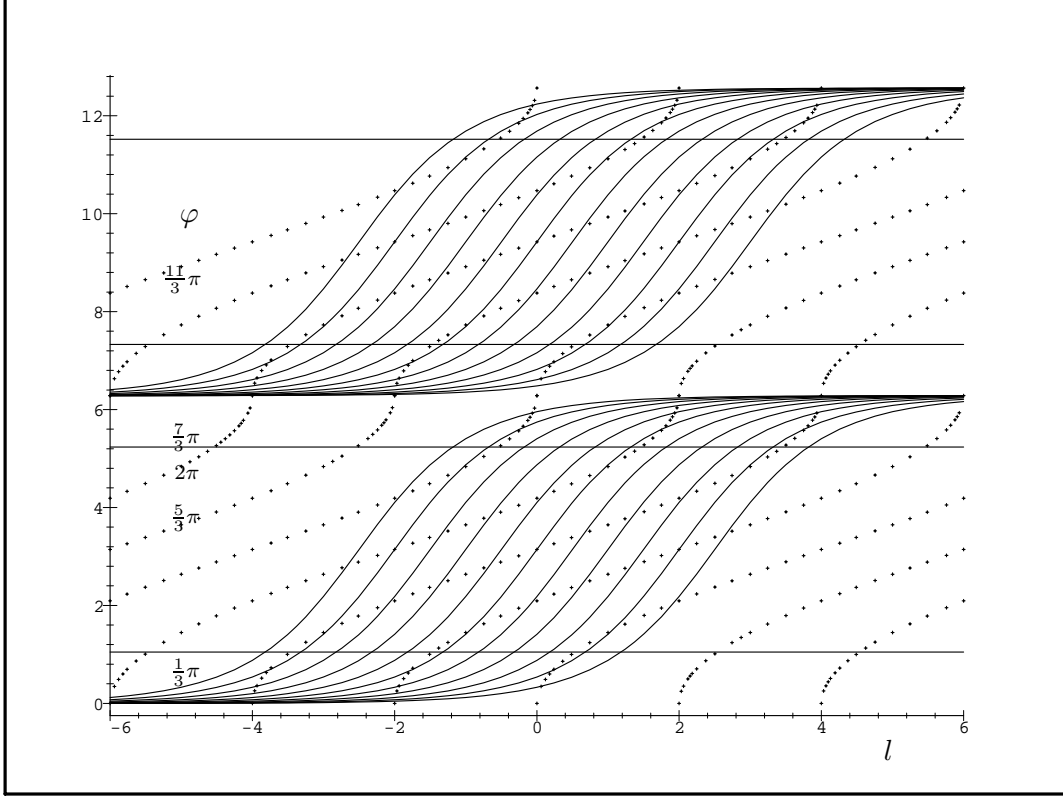
Here we display the 2-dimensional projections of the 2-dimensional extremals ((7.45)) on the plane $\bar{l} = \gamma(t - vx) = \text{const.}$ and those 1-dimensional wave fronts (8.41) that lie entirely in this plane. The “singular” case (8.41) is shown in Fig. 1, the “regular” ones (8.63) for $c_0 = -2$ and $c_0 = 1$ in Figs. 2 and 3, respectively. The extremals are plotted in solid lines, whereas the wave fronts are given in dotted ones. The axes are the field variable and the independent variable $l = \gamma(x - vt)$ which parametrizes the kink solution, obtained from the static one by a Lorentz transformation (see (6.9)).

Fig. 1: The singular case



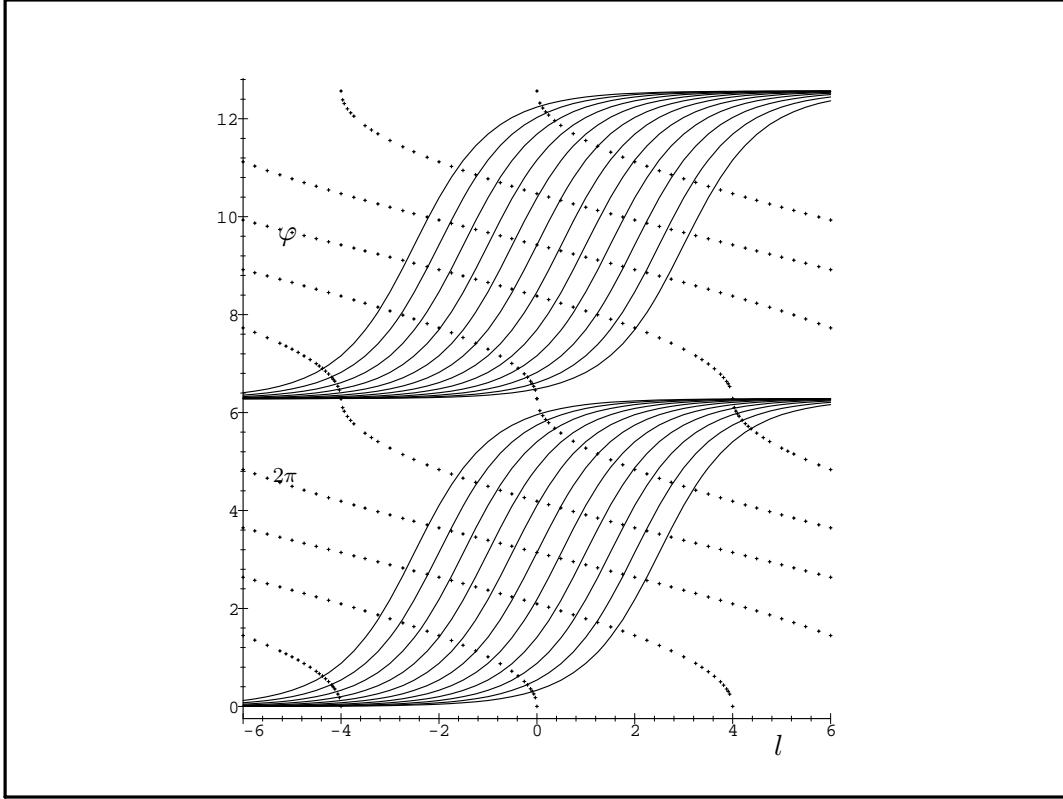
In this case the transversality relations are fulfilled in the extended configuration space \mathcal{M}_{2+1} except on the planes $\varphi = 2k\pi$, $k = 0, \pm 1, \pm 2, \dots$, where the extremals and wave fronts are parallel.

Fig. 2: The regular case for $c_0 = -2$



Here the transversality relations are fulfilled outside the planes $\varphi = 2k\pi \pm \pi/3$, $k = 0, \pm 1, \pm 2, \dots$, where the extremals and wave fronts are parallel since the Lagrangian density vanishes there. Notice, that the transversality relations are fulfilled on the boundaries of the regions $\mathcal{M}_k = \{2k\pi < \varphi < 2(k+1)\pi, z, \bar{z} \in \mathcal{R}\}$, where the families of extremals (7.45) are defined. Therefore the wave fronts can be continued from one of these regions to the next smoothly, contrary to the extremals.

Fig. 3: The regular case for $c_0=1$



Notably, not only the transversality relation is satisfied *everywhere* in the extended configuration space, but in addition the wave fronts intersect the extremals *everywhere* perpendicularly.